Sharp Necessity Gaps for Finite Observations

Albaraa Mesawi

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Abstract

Positive necessity requires a uniform margin below contour one on an assertion's complement. Containing every contour-one parameter does not by itself provide that margin. For inclusive relative-likelihood contours on a finite observation space, I derive a positive gap from the likelihood-envelope masses. For normalized contours the gap bounds every positive necessity value from below, without compactness of the parameter frame or attainment of complementary suprema. The same envelope gives a uniform lower bound on repeated-sampling coverage of the contour-one set, called the core. Both bounds are sharp when the fixed envelope has total mass above one. The gap bounds necessity after a fixed observation, while the coverage bound concerns the sampling probability that the core contains the true parameter. Countably infinite observations permit both zero necessity of the core and zero core coverage, even with attained likelihood maxima and a summable envelope. A finite partition into blocks with globally identical ranks gives sharp grouped-envelope bounds, including for countable observations. For n binomial successes, the core is $[c_n, 1]$, where $c_n = [1 + n^{-1}(1 - 1/n)^{n-1}]^{-1}$. Its width is asymptotic to e^{-1}/n, while its necessity converges to $\exp(-1 - e^{-1})$. The necessity exceeds that limit for every n, whereas the envelope certificates for this observation are at most $1/(n + 1)$. A proof compares every rank interval and identifies the exact necessity for each n. The application clarifies an incidental endpoint statement in a recent possibilistic account of induction. An appendix records the separate domain condition for extending parameter assessments by entailment to additional unknowns.

1 Core containment and positive necessity

A normalized possibility contour assigns parameter scores between zero and one, with supremum one. The possibility of an assertion takes the supremum of the scores inside it. Its necessity subtracts the supremum outside it from one (Liu and Martin, 2021; Martin, 2026). A supremum is the least upper bound; it need not be a score attained by any parameter. Positive necessity therefore requires all excluded scores to stay below one by a common positive margin. The question is when the set of parameters with score one, called the core, suffices to certify this margin and how much necessity can be guaranteed without maximizing over each complementary assertion.

Suppose a hypothetical analyst compares three parameter values with contour scores 1, 1, and 1/4. The assertion allowing only the first parameter has possibility one and necessity zero. The assertion allowing the first two has the same possibility and necessity 3/4. Both assertions include a score of one, but only the second contains every parameter with score one. The scores are contour values. They are not posterior probabilities of the parameters. For an infinite parameter frame, excluded scores can approach one even when each is below one. Core containment then leaves necessity zero.

Propositions 1 and 3 give a sharp envelope bound for normalized inclusive relative-likelihood contours on a finite observation space. The parameter frame may still be infinite. The likelihood

envelope records the supremum of the sampling probabilities of each possible outcome across the model. Its smallest nonobserved mass, divided by its total mass, bounds the necessity of every assertion containing the core. Proposition 2 gives a sharp uniform coverage bound for the random core. Proposition 5 and the following coverage example show why summability of a countable envelope cannot replace finiteness. Proposition 9 gives sharp bounds for finite partitions with globally identical ranks within each block, including countable observation spaces. In the binomial model, Propositions 6 and 7 calculate the all-success core and its necessity for every sample size. Proposition 8 shows that, for this observation, the envelope bound is at most $1/(n+1)$, while the exact necessity stays above $\exp(-1-e^{-1})$, which exceeds one quarter.

Write δ_x for this ratio of envelope masses. The gap has three uses. After observing x, it certifies necessity at least δ_x for every assertion that contains the core; the certificate needs only the $|\mathcal{X}|$ envelope masses and no maximization over complementary assertions. It also excludes necessity values in $(0, \delta_x)$, so a computed value in that range signals an error in the computation or a violated assumption. Finally, it measures how much envelope information alone can certify, and the binomial comparison shows that this amount can be a vanishing fraction of the exact necessity as n grows. When every likelihood maximum is attained, the certificate is the smallest probability that the normalized maximum likelihood distribution (Grünwald, 2004) assigns to a nonobserved outcome. It is sharp over models with the same envelope when the envelope's total mass exceeds one. Martin (2019) observes that plausibility contours built from random sets for discrete data have plateaus and corners that are not yet well understood. For the inclusive relative-likelihood contour on a finite observation space, the top plateau is the core. The results below bound its necessity and coverage, and the binomial application calculates the core and its necessity exactly.

Martin (2019) studies calibration of belief across assertions, and Cella and Martin (2022) distinguish prediction-region coverage from calibration for general prediction assertions. Their validity conditions control the sampling probability of misleading assessments. I take the inclusive contour construction as given and quantify its positive necessity values. The coverage result links the finite gap to its established calibration property; the observation-specific necessity certificate and the uniform coverage coefficient remain distinct.

Liu and Martin (2021) define a possibility measure on subsets of a frame by taking the supremum of a normalized contour over the points in each subset. They define necessity by conjugacy. The later review by Martin (2026) uses the same supremum construction for parameter events and the same conjugacy rule. Hose (2022) also presents the inclusive relative-likelihood transform and its binomial instance. The standard level-set inequality presented there assigns necessity at least α to $\{\theta : \pi_x(\theta) > 1-\alpha\}$ (Hose, 2022). My finite-observation result identifies an explicit $\alpha = \delta_x > 0$, calculated from the likelihood envelope, for which this strict level set equals the core. The contribution is this level and its sharpness; the contour construction and general level-set inequality are established results.

The possibilistic account of induction by Martin, Prim, Raner, and Williams (2026), called the induction account below, claims that an assertion has positive necessity exactly when every excluded parameter has contour below one. It also gives an all-success binomial core and extends the parameter contour by entailment to a larger proposition class. I use the finite and countable results to determine the criterion's scope, then calculate the binomial core and its necessity. Appendix B examines the separate domain condition for the entailment extension.

The account assesses propositions involving the observed evidence with that evidence fixed (Martin, Prim, Raner, and Williams, 2026). I accept this conditional interpretation.

2 From core containment to a uniform margin

On a parameter frame Θ, let π_x be normalized by $\sup_\theta \pi_x(\theta) = 1$, and write

$$\Pi_x(H) = \sup_{\theta \in H} \pi_x(\theta), \qquad N_x(H) = 1 - \sup_{\theta \notin H} \pi_x(\theta), \qquad C_1(x) = \{\theta : \pi_x(\theta) = 1\}. \tag{1}$$

The set $C_1(x)$, called the core, contains the parameters receiving score one. Conjugacy gives the exact condition

$$N_x(H) > 0 \quad \Longleftrightarrow \quad \sup_{\theta \notin H} \pi_x(\theta) < 1. \tag{2}$$

If the complementary supremum is attained, this is equivalent to every complementary contour value being below one, or $C_1(x) \subseteq H$. I use $\sup \varnothing = 0$, so the full frame has necessity one.

The pointwise condition excludes a score of one without bounding how closely excluded scores approach one. The induction account's positive-necessity criterion uses the pointwise condition. Its proof equates a maximum below one with every term being below one (Martin, Prim, Raner, and Williams, 2026). Attainment licenses that equivalence. For a supremum, only the forward implication is automatic. The distinction matters because the construction expressly allows finite or countable sample spaces (Martin, Prim, Raner, and Williams, 2026).

2.1 Inclusive ranking and normalization

The sampling construction imposes more structure than an arbitrary normalized contour. For a finite or countable sample space $\mathcal{X}$, let $p_\theta(y)$ be sampling probabilities. Discard outcomes impossible under every parameter, and put

$$\begin{aligned} d_y &= \sup_{\vartheta \in \Theta} p_\vartheta(y) > 0, \qquad R(y,\theta) = p_\theta(y)/d_y, \\ \pi_x(\theta) &= \sum_{y \in \mathcal{X}} p_\theta(y) \mathbf{1}\{R(y,\theta) \le R(x,\theta)\}. \end{aligned} \tag{3}$$

For each fixed θ, the contour sums the sampling mass of outcomes whose relative-likelihood rank is no higher than that of the observed outcome x. The comparison uses the same θ throughout, and equality retains ties. The observed outcome is always included. A score below one must therefore omit some other outcome with positive sampling mass.

This is the induction account's inclusive ranking construction (Martin, Prim, Raner, and Williams, 2026), with suprema used when likelihood maxima do not exist. Normalization remains a separate requirement whenever (1) is interpreted as possibility and necessity. An attained likelihood maximizer at the observed x suffices: it has rank one and contour one. Every model used below for necessity assessments has such a maximizer.

Supremum denominators alone do not suffice. For $t \in (0,1)$, take probabilities $(t/4, (1+t)/4, (3-2t)/4)$ on $\{x,y,z\}$. The denominators are $(1/4, 1/2, 3/4)$. After observing x, the contour is $t/4$ for $t < 3/5$ and $(3-t)/4$ for $t \ge 3/5$, with supremum $3/5$. Its formal conjugate gives the empty event value $2/5$ and is not a necessity measure. The numerical bound below also applies to this unnormalized score, but its necessity interpretation does not.

For a concrete case, suppose a hypothetical analyst observes one of three labels x, y, z and allows every probability distribution over them. After observing x, the candidate distribution $(2/5, 1/2, 1/10)$ has the same numbers as its relative-likelihood ranks, since all three envelope masses are one. The contour includes the masses at x and z and equals $1/2$. The omitted outcome y has rank $1/2$, above the observed rank $2/5$. This calculation evaluates one candidate. A necessity certificate must bound the supremum of the excluded scores, even when no excluded candidate attains it.

2.2 A sharp bound from the likelihood envelope

The envelope mass d_y records how probable outcome y can be anywhere in the model. A score below one omits at least one outcome from the sum in (3). An envelope mass is an upper bound on that outcome's probability; its positive value alone does not give a lower bound on omitted probability.

The ranking rule supplies the link. The included mass is at most the observed rank multiplied by the sum of the included outcomes' envelope masses, so a contour close to one forces the observed rank to be large. An omitted outcome has a higher rank, so its probability exceeds the observed rank multiplied by its own envelope mass. On a finite observation space, the positive envelope masses of possible omitted outcomes have a positive minimum. The two restrictions then prevent the omitted probability from tending to zero while the contour tends to one, uniformly over parameters.

Proposition 1 (A finite-observation gap)**.** *Suppose* $2 \le |\mathcal{X}| < \infty$*, and fix the observation* x*. Set*

$$S = \sum_y d_y, \qquad m_x = \min_{y \neq x} d_y, \qquad \delta_x = m_x / S > 0.$$

Every value of (3) *below one is strictly less than* $1 - \delta_x$*. If the contour is normalized, then for every event* $H \subseteq \Theta$*,*

$$N_x(H) > 0 \quad \Longleftrightarrow \quad C_1(x) \subseteq H \quad \Longleftrightarrow \quad N_x(H) \ge \delta_x.$$

Proof. Fix θ with $\pi_x(\theta) < 1$ and put $r = R(x, \theta)$. An omitted outcome $y \neq x$ has positive probability and $R(y, \theta) > r$. Its exclusion gives

$$\pi_x(\theta) \le (S - d_y) r, \qquad 1 - \pi_x(\theta) \ge p_\theta(y) > d_y r.$$

Since $S - d_y > 0$, these inequalities imply $1 - \pi_x(\theta) > d_y \pi_x(\theta)/(S - d_y)$, including when $r = 0$. Rearrangement gives $\pi_x(\theta) < 1 - d_y/S \le 1 - \delta_x$. If the complement of H has no contour-one parameter, its supremum is at most $1 - \delta_x$, and $N_x(H) \ge \delta_x$. If it contains a contour-one parameter, its supremum is one and $N_x(H) = 0$. □

When every likelihood maximum is attained, d_y is the maximized likelihood of y, the ratios d_y/S form the normalized maximum likelihood distribution on $\mathcal{X}$, and $\log S$ is the parametric complexity of the model (Grünwald, 2004). The cited source assumes a unique maximizer; uniqueness is not needed here, because only the maximized value enters the definition. With suprema in place of maxima, d_y/S is the direct analogue. For a binomial count the sum over counts equals the sum over sequences, since every sequence with the same count has the same maximized likelihood. The binomial S is therefore the normalizing sum of that definition for the Bernoulli model on sequences of length n. The certificate δ_x is therefore the smallest normalized maximum likelihood probability among nonobserved outcomes, and δ_* in Proposition 2 is the smallest such probability overall. The same source notes that the normalizing sum can be infinite on a countably infinite space. The condition $S < \infty$ in Proposition 5 is the condition for that distribution to exist, and that proposition shows that its existence does not restore the gap.

The parameter frame need not be finite or compact. A single possible observation gives contour one at every parameter, so only necessity values zero and one occur. For two or more observations, Proposition 1 rules out necessity values in $(0, \delta_x)$. It also supports the induction account's criterion for its binomial construction when a complementary maximum does not exist. The strict pointwise bound yields a non-strict necessity bound because a complementary supremum need not be attained.

2.3 A sharp coverage bound for the core

The inclusive ranking construction already supplies the calibration inequality

$$P_\theta\{\pi_X(\theta) \le \alpha\} \le \alpha, \qquad \theta \in \Theta, \quad 0 \le \alpha \le 1, \tag{4}$$

and the strict regions $L_\alpha(x) = \{\theta : \pi_x(\theta) > \alpha\}$ have coverage at least $1 - \alpha$ (Hose, 2022; Liu and Martin, 2021). A direct argument covers finite and countable observations, including ties. For a nonempty finite subset F of $\{x : \pi_x(\theta) \le \alpha\}$, choose an element of F with largest rank. The sampling probability of F is at most the contour at that element and is therefore at most α. Taking the supremum over finite F gives (4). This argument does not require normalization across parameters.

Proposition 2 (Core coverage)**.** *Suppose $2 \le |\mathcal{X}| < \infty$. Put $d_{\min} = \min_y d_y$ and $\delta_* = d_{\min}/S = \min_x \delta_x$. For every parameter,*

$$P_\theta\{\theta \in C_1(X)\} > \delta_*.$$

If $S > 1$, the infimum of these coverage probabilities over the capped-simplex model of Proposition 3 equals δ_*. No uniformly larger coverage bound follows from the envelope alone.

Proof. Proposition 1 gives $C_1(x) = L_{1-\delta_*}(x)$, so (4) gives coverage at least δ_*. For strictness, let M_θ be the nonempty set of outcomes maximizing $R(y, \theta)$, with maximum r. The contour equals one exactly at these observations, and

$$P_\theta\{\theta \in C_1(X)\} = r \sum_{y \in M_\theta} d_y, \qquad 1 = \sum_y d_y R(y, \theta) \le Sr.$$

If M_θ is proper, the last inequality is strict, giving coverage greater than $d_{\min}/S$. If $M_\theta = \mathcal{X}$, the coverage is one. For sharpness, choose j with $d_j = d_{\min}$ in the perturbation of Proposition 3. Its unique largest rank occurs at j, so its coverage is $p_j = d_{\min}/S + \varepsilon$. Let $\varepsilon \downarrow 0$. □

The core therefore forms a confidence region with uniform coverage coefficient δ_*. The observed δ_x instead certifies necessity after x; its data-dependent threshold cannot be inserted into (4) as a fixed confidence coefficient. In the full categorical model on $K \ge 2$ outcomes, $\delta_* = 1/K$, and this uniform coverage floor is sharp. Proposition 8 shows that no envelope on K outcomes gives a larger coefficient δ_*. If $S = 1$, every sampling distribution equals the envelope and the core covers with probability one. Contour normalization remains necessary only for interpreting the conjugate assessments as necessity measures.

2.4 Sharpness from envelope information

The lower bound uses only the envelope. A particular model or assertion can have larger necessity, so Proposition 1 does not calculate every assessment exactly. To determine whether a uniformly larger bound follows from the same information, the next construction holds the envelope fixed and lets excluded scores approach the limit.

Proposition 3 (Sharpness from envelope information)**.** *Fix a finite observation space with at least two points and numbers $0 < d_y \le 1$ with $S = \sum_y d_y > 1$. No number larger than δ_x is a lower bound for $N_x(H)$ over all models with these likelihood-envelope masses and all events $H \supseteq C_1(x)$: a normalized model with these masses attains $N_x(C_1(x)) = \delta_x$.*

Proof. Take as parameter frame all probability vectors p satisfying $p_y \le d_y$. Each coordinate attains d_y: the remaining mass $1 - d_y$ fits within the other coordinates, whose total capacity is $S - d_y \ge 1 - d_y$. These are exactly the model's envelope masses, and a likelihood maximizer at x exists.

Choose $j \ne x$ with $d_j = m_x$. For sufficiently small $\varepsilon > 0$, the vector

$$p_j = \frac{d_j}{S} + \varepsilon, \qquad p_i = \frac{d_i}{S} - \frac{\varepsilon d_i}{S - d_j} \quad (i \ne j)$$

belongs to the frame. Its ranks at all $i \ne j$ equal $1/S - \varepsilon/(S - d_j)$, and its rank at j is larger. Only j is omitted after observing x, so

$$\pi_x(p) = 1 - \frac{d_j}{S} - \varepsilon \longrightarrow 1 - \delta_x.$$

These parameters lie outside $C_1(x)$. Proposition 1 gives the reverse supremum bound there, which proves $N_x(C_1(x)) = \delta_x$. □

When $S = 1$, every distribution with envelope d equals d, so all contours are one. For the full categorical model on $K \ge 2$ outcomes, every $d_y = 1$. The sharp lower bound is $1/K$: an assertion containing the contour-one set receives necessity at least $1/K$, and the core itself receives exactly $1/K$. This statement quantifies support from the core without maximizing over each complementary assertion. It concerns inference from one categorical observation, or any finite observation alphabet to which (3) applies.

Returning to the three-label model, the core consists of distributions in which x has probability at least as large as either other label. That assertion has necessity exactly $1/3$. Distributions $(1/3 - \varepsilon/2, 1/3 + \varepsilon, 1/3 - \varepsilon/2)$, with $0 < \varepsilon < 2/3$, lie outside the core and have contour $2/3 - \varepsilon$. They approach the bound without attaining it outside the core.

2.5 Why a countable observation space changes the conclusion

The proof of Proposition 1 needs a positive minimum among the nonobserved envelope masses. Countably many such masses can tend to zero. The next model shows that a finite sum of envelope masses does not extend the guarantee to a countably infinite observation space.

Example 4 (A missing complementary maximum)**.** *Take* $\Theta = \{0, 1, 2, \ldots\}$ *and* $\mathcal{X} = \{x, y_1, y_2, \ldots\}$. *Define*

$$p_0(x) = 1, \qquad p_k(x) = 1 - 2^{-k}, \qquad p_k(y_k) = 2^{-k} \quad (k \ge 1),$$

with all other masses zero. Each row is a probability distribution. The likelihood denominators are $d_x = 1$ *and* $d_{y_k} = 2^{-k}$*, attained at parameters* 0 *and* k*, respectively, and their sum is two. After observing* x*, parameter* $k \ge 1$ *gives rank* $1 - 2^{-k}$ *to* x *and rank one to* y_k*. Only the mass at* x *contributes, and*

$$\pi_x(0) = 1, \qquad \pi_x(k) = 1 - 2^{-k} \quad (k \ge 1).$$

Let $H = \{0\}$*. Every parameter outside* H *has contour below one, yet its complementary supremum is one. Therefore* $C_1(x) = H$ *and* $N_x(H) = 0$.

In this hypothetical model, an analyst can read x as a routine record and y_k as an alternative record that occurs only under parameter k. Its probability 2^{-k} becomes smaller as k increases. If the model contained only parameters $0, 1, 2, 3$ and outcomes x, y_1, y_2, y_3, the same sampling rows would retain denominators $1, 1/2, 1/4, 1/8$. The largest excluded score would be $7/8$, giving

$N_x(\{0\}) = 1/8$. Proposition 1 gives the uniform bound 1/15 for this truncated model. The infinite model includes alternatives with scores above every fixed threshold below one. Each excluded parameter has a positive shortfall, but those shortfalls have no common positive lower bound.

Proposition 5 (Countable dichotomy)**.** *Let the observation space have at least two outcomes and be finite or countably infinite, with $S = \sum_y d_y < \infty$, and fix x. Every value of* (3) *below one is strictly less than $1 - \delta_x$, where $\delta_x = \inf_{y \neq x} d_y/S$. If the space is finite, $\delta_x > 0$. If the space is countably infinite, then $\delta_x = 0$, and the model whose parameter frame consists of all probability vectors p with $p_y \leq d_y$ for every y, for any such envelope with $S > 1$, has excluded contours with supremum one, so $N_x(C_1(x)) = 0$.*

Proof. The proof of Proposition 1 uses only $S < \infty$: for an omitted y, $\pi_x(\theta) \leq (S - d_y)r$ and $1 - \pi_x(\theta) > d_y r$, which give $\pi_x(\theta) < 1 - d_y/S \leq 1 - \delta_x$. Countably many positive masses with a finite sum have infimum zero. In the capped-simplex model, the vector of Proposition 3 with j in place of the minimizing index has contour $1 - d_j/S - \varepsilon$ and lies outside the core; letting j run through the space gives supremum one. □

Example 4 has the envelope $d_x = 1$ and $d_{y_k} = 2^{-k}$, with $S = 2$. Its frame is a countable subset of the capped simplex with that envelope, and it already attains $N_x(C_1(x)) = 0$ on that smaller frame. The four-outcome truncation discussed after the example has the positive envelope gap 1/15, while the infinite model has none.

Countably infinite observations can also give zero core coverage. Take the capped-simplex model on $i = 0, 1, \ldots$ with $d_i = 2^{-i}$ and $S = 2$. Every coordinate likelihood maximum exists, so each observed contour is normalized. The model contains the parameter

$$p_i^* = 2^{-i}\left(\frac{2}{3} - \frac{2^{-i}}{4}\right), \qquad \sum_{i=0}^{\infty} p_i^* = \frac{4}{3} - \frac{1}{3} = 1.$$

Its ranks increase strictly to 2/3. For every observed i, the contour at p^* is $\sum_{j=0}^{i} p_j^* < 1$. The true parameter therefore lies outside the core for every observation, although the fixed-level regions retain (4). Finiteness supplies both the positive necessity gap and a positive uniform core coverage floor.

Figure 1 compares the complementary suprema in the three-label and countable models. Panel (a) uses a continuous slice of the parameter frame for a finite observation space. Panel (b) shows discrete parameters in the countable model; the formula for all $k \geq 1$ establishes its infinite limit.

The example uses the induction account's inclusive relative-likelihood construction. All elementary likelihood maxima exist, and the global contour maximum is attained. The only missing maximum is the one over the complementary assertion. If countability is also imposed on the proposition class, the finite/cofinite Boolean algebra on Θ contains the required singletons, H, and H^c and remains countable. The example requires no uncountable proposition class.

With the induction account's literal maximum notation, the upper assessment of H^c is undefined. With the standard supremum extension it is defined, and the pointwise criterion is false. An explicit assumption that every relevant complementary maximum exists excludes this example and validates the criterion. Since H is elementary and the proposition class is closed under negation, that assumption restricts admissible models. Proposition 1 gives a different sufficient condition. The general repair is (2).

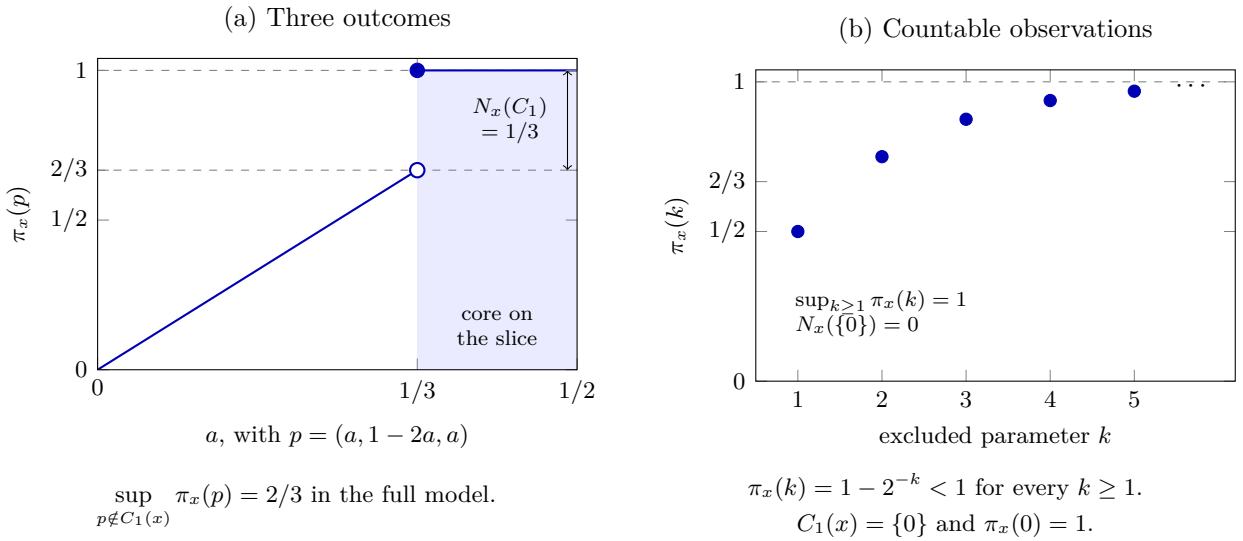


Figure 1: Inclusive relative-likelihood contours after observing x. (a) In the full three-outcome categorical model, the slice $p = (a, 1-2a, a)$ has contour $2a$ below $a = 1/3$ and contour one from that point onward. Inclusive ties produce the filled endpoint at one. Excluded parameters on this slice approach $2/3$; Proposition 1 bounds every excluded score in the full model by $2/3$. The full model's complementary supremum is therefore $2/3$, and its core has necessity $1/3$. (b) In the countable model of Example 4, excluded parameter k has contour $1 - 2^{-k}$. These scores approach one without reaching it, so the core $\{0\}$ has necessity zero.

3 Application: the all-success binomial core

The finite-observation result makes the core decisive for positive necessity. I now calculate both the core and its exact necessity for an all-success binomial observation. The calculation also clarifies the induction account's incidental endpoint statement: after ten successes, $H = [19/20, 1]$ has positive necessity under the inclusive contour, whereas the printed core would give it necessity zero. The account's conclusion that some assertions receive positive necessity survives.

The induction account states that after n successes in n Bernoulli trials the contour equals one on an interval $[c, 1]$ and adds, in a remark described there as incidental, that $c = (n-1)/n$ (Martin, Prim, Raner, and Williams, 2026). Its binomial formula uses the inclusive ranking in (3) (Martin, Prim, Raner, and Williams, 2026), and its Figure 1 reports possibility approximately 0.42 for the interval $[0.7, 0.9]$ at $n = 10$, a value consistent with the boundary derived below (the inclusive contour at $\theta = 0.9$ equals 0.4189) and inconsistent with a plateau beginning at 0.9. The stated endpoint and the formula disagree; the formula is the construction.

Proposition 6. *For $X \sim \mathrm{Bin}(n, \theta)$ with $\Theta = [0, 1]$, positive integer n, and observation $X = n$, equation (3) gives*

$$C_1(n) = [c_n, 1], \qquad c_n = \left[1 + \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1}\right]^{-1} \qquad (n \geq 1).$$

Proof. For $n = 1$, the ranks are $R(1, \theta) = \theta$ and $R(0, \theta) = 1 - \theta$. The contour after a success is θ for $\theta < 1/2$ and one for $\theta \geq 1/2$. With $0^0 = 1$, the formula returns $c_1 = 1/2$.

The likelihood of count k attains its maximum at $\theta = k/n$, so its envelope mass is

$$d_k = \binom{n}{k}(k/n)^k(1-k/n)^{n-k}, \tag{5}$$

with $0^0 = 1$ at the endpoints. For $n \geq 2$ and $0 < \theta < 1$, every count has positive probability. The contour is one exactly when $R(k,\theta) \leq R(n,\theta) = \theta^n$ for all $k < n$. Put $q = (1-\theta)/\theta$. The condition for $k = 0$ is $q \leq 1$. For $0 < k < n$, set $t = k/n$. Cancelling the binomial coefficient gives

$$\frac{R(k,\theta)}{R(n,\theta)} = \frac{q^{n-k}}{t^k(1-t)^{n-k}},$$
$$R(k,\theta) \leq \theta^n \quad \Longleftrightarrow \quad q^{n-k} \leq t^k(1-t)^{n-k}.$$

Taking the $(n-k)$th root gives

$$q \leq s(t) := (1-t)t^{t/(1-t)}.$$

The function tends to one as $t \downarrow 0$, and

$$\frac{d}{dt}\log s(t) = \frac{\log t}{(1-t)^2} < 0 \qquad (0 < t < 1).$$

The most restrictive count is $k = n-1$. Its bound is

$$q \leq \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1},$$

equivalent to $\theta \geq c_n$. Inclusive ties include the endpoint. At $\theta = 0$ the observed contour is zero, and at $\theta = 1$ it is one. □

For $n = 2$, $c_2 = 4/5$. At the stated boundary $\theta = 1/2$, the count ranks are $(1/4, 1, 1/4)$ and the contour after observing two successes is $1/2$. For $n = 10$, the corrected boundary is approximately 0.962703, rather than 0.9.

By Proposition 1, an event has positive necessity exactly when it contains $[c_n, 1]$. This corrects the necessary boundary without invalidating the induction account's sufficient examples: for $n \geq 2$, $c_n > (n-1)/n$, so an interval containing $[(n-1)/n, 1]$ also contains the actual core. The same implication holds for $n = 1$. The conclusion that some assertions obtain positive necessity survives. The printed plateau endpoint does not.

The correction also identifies assertions that the printed plateau would exclude. For $n = 10$, the event $H = [19/20, 1]$ contains the actual core and therefore has $N_{10}(H) \geq \delta_{10} > 0$. A plateau extending down to $9/10$ would instead force $N_{10}(H) = 0$. The envelope in (5) gives $d_5 = 63/256$ and $\delta_{10} \approx 0.053$. Since $\delta_* = d_{\min}/S = \min_x \delta_x$ equals this δ_{10}, Proposition 2 also gives uniform core coverage at least 0.053. Proposition 8 shows that δ_n and δ_* are at most $1/(n+1)$ for every n. The next result calculates the exact core necessity, and Proposition 8 shows that it stays above one quarter.

Proposition 7 (Exact binomial core necessity)**.** *For every positive integer n, the all-success binomial contour has*

$$N_n(C_1(n)) = nc_n^{n-1}(1-c_n) = b_n(1+b_n/n)^{-n}, \qquad b_n = (1-1/n)^{n-1}, \tag{6}$$

with $b_1 = 1$. This necessity converges to $\exp(-1-e^{-1})$.

Proof. For $n = 1$, the contour in the preceding proof gives necessity $1/2$, as in (6). Suppose $n \geq 2$. Put $Y = n - X$, $u = 1 - \theta$, and $q = u/(1-u)$. Define

$$q_r = \frac{r}{n}\left(1 - \frac{r}{n}\right)^{(n-r)/r}, \qquad a_r = \frac{q_r}{1+q_r} \quad (1 \leq r \leq n), \tag{7}$$

with $q_n = 1$. The preceding rank calculation shows that the thresholds increase strictly, with $a_1 = 1 - c_n$. If $q_r < q \leq q_{r+1}$ for $1 \leq r < n$, exactly the failure counts $1, \ldots, r$ are omitted. Their mass is $F_r(u) = P_u\{1 \leq Y \leq r\}$. Inclusive ties give the open left endpoint. Differentiation gives

$$F_r'(u) = n(1-u)^{n-r-1}\left[(1-u)^r - \binom{n-1}{r}u^r\right] \quad (r < n).$$

The derivative changes sign once, from positive to negative, so the infimum over a rank interval is an endpoint value or a one-sided endpoint limit. At a left endpoint, $F_r(a_r) = F_{r-1}(a_r) + P_{a_r}\{Y = r\}$. Appendix A proves $F_{r-1}(a_r) > 1/3$ for every $2 \leq r \leq n$. For $q > 1$, all positive failure counts are omitted, with mass $1 - (1-u)^n \geq 1 - 2^{-n} \geq 3/4$. At $\theta = 0$, the contour is zero and the omitted mass is one.

On the interval adjacent to the core, only $Y = 1$ is omitted. The bound $a_2 \leq 1/n$, proved in the appendix, makes $P_u\{Y = 1\}$ increasing there. Its infimum is the right limit at a_1,

$$P_{a_1}\{Y = 1\} = nc_n^{n-1}(1 - c_n) = b_n(1 + b_n/n)^{-n} < \frac{1}{3}.$$

For the last inequality, the binomial theorem gives $b_n \leq 1/2$ and $(1 + b_n/n)^n > 1 + b_n$. Every other interval has a larger omitted mass by the endpoint comparison. This proves (6), with the complementary supremum approached as $\theta \uparrow c_n$. Finally, $b_n \to e^{-1}$ and $(1+b_n/n)^n \to \exp(e^{-1})$. □

For $n = 2$, the exact core necessity is $8/25$. For $n = 10$, it is approximately 0.264914, and the excluded contours have supremum approximately 0.735086. The limiting necessity is approximately 0.254646. The core width has the separate limiting scale

$$n(1 - c_n) = \frac{(1 - 1/n)^{n-1}}{1 + n^{-1}(1 - 1/n)^{n-1}} \longrightarrow e^{-1}.$$

The printed width $1/n$ has the same order and a different limiting constant.

The next result compares the envelope certificates with this exact value. Its first part holds for every finite observation space, with S, m_x, δ_x, $d_{\min}$ and δ_* as in Propositions 1 and 2.

Proposition 8 (Envelope certificates against exact binomial necessity)**.** *Let $2 \leq |\mathcal{X}| = K < \infty$. Then $\delta_* \leq 1/K$, with equality exactly when all envelope masses coincide, and $\delta_x < 1/(K-1)$ for every x. In the binomial model of Proposition 6 with observation $X = n$, $\delta_n = \delta_* \leq 1/(n+1)$, while for every positive integer n*

$$N_n(C_1(n)) > \exp(-1 - e^{-1}).$$

Proof. Since $S \geq Kd_{\min}$, $\delta_* = d_{\min}/S \leq 1/K$, with equality exactly when every d_y equals $d_{\min}$. Since $S \geq d_x + (K-1)m_x$ and $d_x > 0$, $S > (K-1)m_x$ and $\delta_x < 1/(K-1)$. In the binomial model, each d_k in (5) is a probability and $d_n = 1$, so d_n is a largest mass and $m_n = \min_{k \neq n} d_k = d_{\min}$. Therefore $\delta_n = \delta_*$, and $K = n + 1$ gives $\delta_* \leq 1/(n+1)$.

For the lower bound, write $N_n(C_1(n)) = b_n(1 + b_n/n)^{-n}$ as in (6). Since $\log(1+y) < y$ for $y > 0$, $(1+b_n/n)^n < e^{b_n}$ and $N_n(C_1(n)) > b_n e^{-b_n}$. For $n \geq 2$, the inequality $\log(1-t) \geq -t/(1-t)$ at $t = 1/n$ gives $(n-1)\log(1 - 1/n) \geq -1$, so $b_n \geq e^{-1}$, and $0 < 1 - 1/n < 1$ gives $b_n \leq 1$. Together with $b_1 = 1$, this gives $e^{-1} \leq b_n \leq 1$ for every n. The function ue^{-u} increases on $[0, 1]$, so $b_n e^{-b_n} \geq e^{-1}\exp(-e^{-1})$. □

The exact core necessity therefore exceeds its limit for every sample size, while the envelope certificate for the all-success observation is at most $1/(n+1)$. Their ratio exceeds $(n+1)\exp(-1-e^{-1})$, which is approximately $0.2546(n+1)$. For $n = 10$ the ratio is approximately 5.02, and for $n = 1000$ it is approximately 407. At $n = 1$ the two outcomes have equal envelope masses, and the certificate $1/2$ equals the exact necessity. As n grows, envelope information alone cannot replace the model-specific calculation.

The induction account also extends its contour to propositions about additional unknowns by entailment. Appendix B states the standard atom condition that guarantees maxitivity for every normalized contour, applies it to an unresolved coordinate, and records a compatibility-based enlargement. This domain condition does not enter the finite-gap or binomial arguments.

4 Grouped observations and scope

For a normalized inclusive relative-likelihood contour on a finite observation space, the core certifies positive necessity with an envelope bound. A uniform threshold also makes the random core a confidence region with an envelope-based coverage floor. Both bounds are sharp over the envelope class when $S > 1$. The necessity level applies after a fixed observation, while the coverage coefficient bounds the sampling probability that the core contains the true parameter. Countably infinite observations admit zero values for both guarantees despite attained likelihood maxima and a summable envelope.

The raw envelope certificate can lose precision under fixed splitting of recorded labels. Known within-block structure gives a stronger bound. The factorization below is the usual sufficient-statistic reduction; Hose (2022) already discusses sufficient statistics for the relative-likelihood transform.

Proposition 9 (Sharp grouped-envelope certificates)**.** *In the setting of* (3)*, let $B_1, \dots, B_K$ be a finite partition of $\mathcal{X}$ into nonempty blocks. Suppose $R(y, \theta) = r_j(\theta)$ for every θ and every $y \in B_j$. Put*

$$D_j = \sum_{y \in B_j} d_y, \qquad q_\theta(j) = \sum_{y \in B_j} p_\theta(y), \qquad S = \sum_{j=1}^{K} D_j.$$

Then $0 < D_j = \sup_\theta q_\theta(j) \le 1$, $S < \infty$, and for every $x \in B_j$ the original and grouped contours agree:

$$\pi_x(\theta) = \sum_{\ell=1}^{K} q_\theta(\ell) \mathbf{1}\{q_\theta(\ell)/D_\ell \le q_\theta(j)/D_j\}.$$

Their cores, possibility and necessity assessments under normalization, and sampling core coverage agree. For $K \ge 2$, set

$$\Delta_j = \frac{\min_{\ell \ne j} D_\ell}{S}, \qquad \Delta_* = \frac{\min_\ell D_\ell}{S}.$$

Every nonunit contour after $x \in B_j$ is strictly below $1 - \Delta_j$. For normalized contours,

$$N_x(H) > 0 \iff C_1(x) \subseteq H \iff N_x(H) \ge \Delta_j.$$

At every parameter, $P_\theta\{\theta \in C_1(X)\} > \Delta_*$. If $S > 1$, the necessity bounds and coverage infimum are sharp among models with the prescribed individual d_y and within-block rank identities.

Proof. Each r_j has supremum one by the definition of d_y. Choose θ with $r_j(\theta) > 0$. Nonnegative summation gives $q_\theta(j) = r_j(\theta) \sum_{y \in B_j} d_y \le 1$, so $D_j < \infty$. Now $q_\theta(j) = D_j r_j(\theta)$, and taking

suprema gives $D_j = \sup_\theta q_\theta(j) \le 1$. Grouping the inclusive contour summands proves the displayed identity. The sampling core event is constant on each block, which proves the preservation claims. Propositions 1 and 2 give the bounds.

For sharpness, use the capped simplex $\{q : 0 \le q_j \le D_j,\ \sum_j q_j = 1\}$ and lift it by $p_\theta(y) = (d_y/D_j)q_\theta(j)$ for $y \in B_j$. Each coordinate q_j attains D_j, since the remaining capacity $S - D_j \ge 1 - D_j$ holds. The fine envelopes are therefore exactly d_y, each observed contour is normalized, and the required rank identities hold. The perturbations in Propositions 3 and 2 give $N_x(C_1(x)) = \Delta_j$ through a complementary supremum and coverage infimum Δ_*. Strict coverage leaves that infimum unattained. □

If $K = 1$ or $S = 1$, every contour is one. Singleton blocks recover the finite-observation theorem. When finite in number, the complete equivalence classes of globally identical rank functions give the largest admissible grouping; a finer admissible partition can weaken the certificate. Arbitrary grouping can change rank comparisons. Equality at one parameter or approximate equality does not suffice, and no method for deciding global functional equality is claimed.

In the full three-label model, replace z by finitely or countably many sublabels with fixed positive weights w_i summing to one, so $p_\theta(z_i) = w_i p_\theta(z)$. Each sublabel has the original z-rank. Grouping them gives $D = (1, 1, 1)$, restoring necessity certificate $1/3$ and coverage infimum $1/3$ without changing any contour. For m equal weights, the raw certificate after x is $1/(3m)$; countably many weights give raw $\delta_x = 0$. Unrestricted envelope-only sharpness remains valid because that larger class drops the splitting constraints. The countable zero-gap examples above have infinitely many distinct rank functions.

5 Conclusion

The all-success binomial calculation gives a model-specific assessment beyond the envelope bound. Its core contracts at rate e^{-1}/n, its necessity converges to $\exp(-1 - e^{-1})$ from above, and the envelope certificate for the same observation is at most $1/(n + 1)$. These properties follow from the inclusive transform and remain valid if the induction account revises its incidental endpoint statement. The proposition-domain check in Appendix B remains separate: the entailment rule preserves binary maxitivity for every normalized contour exactly when each elementary hypothesis is an atom of the enlarged Boolean algebra.

The results do not evaluate the induction account's loss-minimizing estimator or settle the philosophical problem of induction. Hose (2022) presents the transform, the level-set inequality and the binomial instance. Its binomial example plots the contour after nine successes in nine trials; the plateau begins near 0.96, consistent with $c_9 \approx 0.9585$, and the example gives no endpoint formula or necessity value. Martin (2026) presents the transform and its calibration, and Martin, Prim, Raner, and Williams (2026) state the all-success plateau. I found no statement of the envelope level δ_x, its sharpness over a fixed envelope, or the endpoint c_n in the main text of Martin (2026), in Martin, Prim, Raner, and Williams (2026), or in the sections of Hose (2022) that present the transform, the level-set inequality and the binomial example. I make no priority claim beyond these comparisons. The sampling result is a sharp consequence of the transform's established calibration, rather than a new general calibration principle. Continuous observation models require a different analysis because an omitted outcome can have zero probability, so the omitted-mass argument used here does not apply.

A Binomial endpoint comparison

Lemma 10. *For $2 \le r \le n$ and $Y \sim \mathrm{Bin}(n, a_r)$ with (7),*

$$P\{1 \le Y \le r-1\} > 1/3.$$

Also, $a_r \le r/(2n)$ for $1 \le r \le n$.

Proof. Write $v = r/n$ and $h = (1-v)^{(1-v)/v}$, with $h = 1$ at $v = 1$. Then $q_r = vh$. The inequality $\log(1-v) \ge -v/(1-v)$ gives $h \ge e^{-1} > 1/3$. For $z = 1-v$, weighted arithmetic-geometric mean gives

$$z^z(1+z)^{1-z} \le z^2 + (1-z)(1+z) = 1.$$

It follows that $h \le (2-v)^{-1}$ and $a_r = vh/(1+vh) \le v/2$.

For $r \ge 4$, concavity of $\log(1+v/3)$ gives

$$P(Y=0) = (1+q_r)^{-n} \le (1+v/3)^{-n} \le (3/4)^r.$$

The second factorial moment gives

$$P(Y \ge r) \le \frac{n(n-1)a_r^2}{r(r-1)} \le \frac{r}{4(r-1)}.$$

Both upper bounds decrease with r. Therefore

$$P(1 \le Y \le r-1) \ge 1 - (3/4)^r - \frac{r}{4(r-1)} \ge \frac{269}{768} > \frac{1}{3}.$$

For $r = 2$, $a_2 \le 1/n$ and $q_2 \ge 2/(3n)$. The probability of one failure increases up to $u = 1/n$, so

$$P(Y=1) \ge \frac{2/3}{(1+2/(3n))^n} \ge \frac{2}{3}e^{-2/3} > \frac{1}{3}.$$

For $r = 3$, put $\lambda = na_3$. The bounds above give $3/4 < 3/(e+1) \le \lambda \le 3/2$. The elementary inequality $\log(1-t) \ge -t - t^2/[2(1-t)]$ gives $P(Y=1) \ge \lambda e^{-\lambda}$: its exponent bound holds for $\lambda \le 2n/(n+1)$, which follows from $n \ge 3$ and $\lambda \le 3/2$. Also,

$$\frac{P(Y=2)}{P(Y=1)} = \frac{(n-1)\lambda}{2(n-\lambda)} \ge \frac{\lambda}{3}.$$

The function $\lambda(1+\lambda/3)e^{-\lambda}$ has one positive critical point, a maximum. It follows that

$$P(1 \le Y \le 2) \ge \lambda(1+\lambda/3)e^{-\lambda} \ge \min\left\{\frac{15}{16}e^{-3/4}, \frac{9}{4}e^{-3/2}\right\} > \frac{3}{8} > \frac{1}{3}.$$

These cases exhaust $2 \le r \le n$. □

B The entailment extension and additional unknowns

The preceding results assess subsets of the parameter frame. Applying their upper and lower interpretation to propositions about additional unknowns requires an extension to a larger domain. The induction account's entailment rule attempts such an extension. The check is therefore part

of the application: a correct parameter contour cannot repair a failure of the extension's possibility axioms. I now record when that rule guarantees binary maxitivity, the requirement that a disjunction's possibility equals the larger of its two possibilities. The characterization is a standard consequence of the extension principle on a Boolean algebra and is stated here only to make the domain condition explicit; it does not enter the finite-observation proof.

Let $\mathcal{B}$ be a Boolean algebra ordered by entailment, and let $\mathcal{E}$ be a nonempty subset of $\mathcal{B} \setminus \{\bot\}$. A normalized contour $\pi : \mathcal{E} \to [0,1]$ satisfies $\sup_{\eta \in \mathcal{E}} \pi(\eta) = 1$. Consider

$$\Pi_\pi^{\subseteq}(A) = \sup\{\pi(\eta) : \eta \in \mathcal{E},\ \eta \leq A\}, \qquad \sup \varnothing = 0. \tag{8}$$

A supremum makes the extension defined without an attainment assumption. Binary maxitivity requires

$$\Pi(A \vee D) = \max\{\Pi(A), \Pi(D)\}, \qquad \Pi(\bot) = 0, \quad \Pi(\top) = 1. \tag{9}$$

A nonzero element η is prime when $\eta \leq A \vee D$ implies $\eta \leq A$ or $\eta \leq D$ for every $A, D \in \mathcal{B}$.

An atom is a nonzero proposition with no nonzero proposition strictly below it. Within the chosen algebra, it leaves no finer alternative unresolved. The next result explains why the choice of algebra matters: an elementary hypothesis can be an atom among parameter events and cease to be one after additional unknowns enter the domain.

Proposition 11. *Equation* (8) *satisfies* (9) *for every normalized contour on $\mathcal{E}$ if and only if every member of $\mathcal{E}$ is an atom of $\mathcal{B}$.*

Proof. If every member is prime, the elementary hypotheses entailing $A \vee D$ form the union of those entailing its disjuncts. Taking a supremum gives maxitivity. Nonzero hypotheses do not entail $\bot$, and normalization gives the value at $\top$.

If η_0 is not prime, choose A, D with $\eta_0 \leq A \vee D$, $\eta_0 \not\leq A$, and $\eta_0 \not\leq D$. Assign contour one to η_0 and zero to every other elementary hypothesis. Equation (8) gives one to $A \vee D$ and zero to both disjuncts.

In a Boolean algebra, a nonzero prime element is an atom. Primeness applied to $\beta \vee \neg\beta = \top$ shows that a prime element η satisfies $\eta \leq \beta$ or $\eta \leq \neg\beta$ for every β. If $\bot < \beta < \eta$, the first alternative contradicts $\beta < \eta$ and the second gives $\beta \leq \eta \leq \neg\beta$, so $\beta = \bot$. Conversely, an atom decides every proposition because its meets with A and $\neg A$ partition it. □

The quantifier over contours cannot be dropped. On worlds $\{a, b\}$, include all three nonempty subsets as elementary hypotheses and assign each contour one. Equation (8) is maxitive, although $\{a, b\}$ is not an atom. Singleton hypotheses already give one to every nonempty event. The proposition states a condition for a guarantee over all contours.

Example 12 (An unresolved coordinate)**.** *Let $\Omega = T \times \{0,1\}$ for a nonempty set T, let $\eta_t = \{t\} \times \{0,1\}$, and let $A = T \times \{1\}$. No elementary hypothesis entails A or A^c. Every normalized contour on these cylinders gives*

$$\Pi^{\subseteq}(A) = \Pi^{\subseteq}(A^c) = 0, \qquad \Pi^{\subseteq}(\Omega) = 1.$$

Maxitivity fails. The formal conjugate $1 - \Pi^{\subseteq}(A^c)$ gives one to both complementary propositions and exceeds their upper scores.

The induction account defines its proposition class as closed under negation and conjunction (Martin, Prim, Raner, and Williams, 2026). Its swan/cat example says that elementary hypotheses specify proportions of white swans and that a hypothesis about cat color cannot be constructed

from them (Martin, Prim, Raner, and Williams, 2026). The entailment extension assigns zero when no elementary hypothesis entails the proposition. In the account's parameter-event domain, the power set of $[0, 1]$ with point hypotheses (Martin, Prim, Raner, and Williams, 2026), every nonempty event has an elementary entailer. A nonempty proposition with no elementary entailer therefore requires a larger domain. This differs from an event within the parameter-event domain whose elementary entailers all have score zero; that event also receives possibility zero. The cat-color proposition requires a larger domain. If the swan hypotheses leave both cat-color alternatives unresolved, Example 12 applies. Closure under negation places both alternatives in the assessed domain. The upper and lower probability interpretation then fails on that enlarged domain.

Restricting the domain to parameter events repairs this failure. Point parameter hypotheses are atoms of the parameter powerset algebra, so the usual pointwise extension works. That restriction excludes the cat-color example. The account's observed-evidence slice addresses propositions involving the observed data; it does not supply information about a separate unknown coordinate. A cat proposition outside the intended domain receives no assessment under the entailment extension.

One explicit enlargement uses compatibility. The nonempty-overlap criterion has an antecedent in the multivalued mapping of Dempster (1967). Dempster aggregates a probability measure over compatible preimages; here I take a supremum of possibility scores. Let each coarse hypothesis have a nonempty set $\Gamma(\eta) \subseteq \Omega$ of compatible complete worlds, and define

$$\Pi^*(A) = \sup\{\pi(\eta) : \Gamma(\eta) \cap A \neq \varnothing\}. \tag{10}$$

This is the ordinary pointwise construction with world contour

$$\pi^*(\omega) = \sup\{\pi(\eta) : \omega \in \Gamma(\eta)\}.$$

The two suprema run over the same pairs (η, ω). Nonempty fibers give normalization, and compatibility with a union is compatibility with at least one member. Equation (10) is therefore maxitive. In Example 12 it assigns possibility one and necessity zero to both alternatives. The disjoint-cylinder representation also preserves every parameter-event assessment: for $H \subseteq T$ and $\Gamma(\eta_t) = \{t\} \times \{0, 1\}$,

$$\Pi^*(H \times \{0, 1\}) = \sup_{t \in H} \pi(\eta_t),$$

and conjugacy preserves necessity. With arbitrary overlapping fibers, the construction defines a transfer that can raise a coarse hypothesis's score, so it need not preserve the original assessments. Other joint information can support other enlargements; uniqueness is not claimed.

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