Critical Note

Interpreting the reported mean acceleration

Comment on "Estimating Flight Characteristics of Anomalous Unidentified Aerial Vehicles" (Knuth et al., 2019)

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Abstract

Knuth et al. report a mean acceleration near $550g_0$ for their visual-departure calculation. Their summary table uses the most probable value near $150g_0$. This note examines the meaning and reproducibility of the reported mean. Reconstructing their Figure 4D acceleration histogram gives $548.5g_0$. The analysis interprets the stated Gaussian values as parameters before truncation, adds an angular bound that keeps distance positive, and assumes angle-duration independence. The resulting model has an infinite unrestricted mean because its duration density approaches a positive value at zero and acceleration depends on inverse-square duration. A finite simulation run produces a finite average almost surely; conditioning on a bounded acceleration range gives a finite mean. Applying these choices to the reported inputs and restricting acceleration to $2000g_0$ gives a mean of $595.5g_0$ and excludes 27.4% of the model's probability. A separate fit to their Figure 4C timing bars gives a restricted mean of $549.9g_0$, close to the reported number, without identifying their averaging procedure. The analysis distinguishes finite interpretations of the reported number and identifies the information needed to reproduce it. Duration models with identical means and variances can imply different acceleration expectations; the Bethune calculation shares the inverse-square duration dependence. The analysis leaves their modal summary and separate encounter estimates unchanged.

1 Background and the reported mean

Knuth et al. (2019) estimate flight characteristics from reports of unidentified aerial vehicle encounters. Their calculations combine reported distances and durations with kinematic assumptions and probability distributions representing uncertainty. Section 2.4.2 examines the visual departure described by Commander David Fravor during the 2004 Nimitz encounter. The authors assume constant acceleration and infer the distance traveled from an assumed object width and the angular threshold at which it would disappear from view. They report a most probable acceleration near $150g_0$ and a mean near $550g_0$ in Figure 4; Table 2 uses $150^{+140}_{-80}g_0$. Here $g_0 = 9.80665\ \mathrm{m\,s^{-2}}$.

The mean requires particular care because the inferred acceleration increases as the inverse square of duration. If the reported 1 ± 1 s notation is interpreted as Gaussian location and scale before positive-time truncation, the resulting distribution assigns probability to arbitrarily short positive times. Durations near zero can dominate an average even when the most probable acceleration remains finite. Knuth et al. discuss transformations of probability densities and define means by expectation integrals in Section 2.2. This note examines whether the finite reported average can represent that unrestricted expectation, how closely their Figure 4D histogram reproduces the reported number, and which information would specify a reproducible finite alternative.

2 Model and statistical distinctions

Knuth et al.'s visual-departure calculation uses

$$A = \frac{2D}{T^2}, \qquad D = \frac{w/2}{\tan(\theta/2)}. \tag{1}$$

Here D is the distance traveled, T is duration, w is object width and θ is the angular visibility threshold. The calculation assumes negligible initial speed. Knuth et al. assign $w = 10$ ft, or 3.048 m, and visual acuity of 2 ± 1 arcmin with a lower cutoff of 1 arcmin. They assign a duration of 1 ± 1 s, truncated to positive time (Knuth et al. 2019, sec. 2.4.2). This note treats the stated Gaussian values as location and scale before truncation, adds $\theta < \pi$ to keep distance positive, and assumes angle-duration independence for numerical propagation. Together with Knuth et al.'s stated inputs, these choices define the independent model.

Knuth et al. describe constant acceleration as a lower-bound model and note that brief acceleration followed by coasting could require a larger acceleration (Knuth et al. 2019, sec. 2.4.2). That statement concerns the assumed motion. The model mean is the expectation of acceleration under the assigned uncertainty distribution. A mode maximizes the acceleration density; a median has half the probability on either side. A distribution can have a finite mode and median while having an infinite mean.

Three averages require separate definitions. An unrestricted model mean includes the entire assigned probability distribution. A finite simulation average includes the values drawn in a particular run. A restricted mean includes only a stated acceleration range and normalizes by the probability retained in that range. The Figure 1 caption in Knuth et al. identifies their plots, including Figure 4, as sample-count histograms (Knuth et al. 2019). The bar reconstruction uses only the counts represented in Figure 4D. The authors do not specify whether they averaged only the samples represented in Figure 4D.

At fixed distance, halving the duration quadruples the inferred acceleration. The positive-time Gaussian formed from the stated 1 ± 1 s timing values has pre-truncation location $\mu = 1$ s, scale $\sigma = 1$ s, and a duration density f_T whose limit at zero is strictly positive, $f_T(0+) > 0$. Assuming angle and duration are independent,

$$\mathbb{E}[A] = 2\mathbb{E}[D] \int_0^\infty \frac{f_T(t)}{t^2}\,\mathrm{d}t = \infty. \tag{2}$$

The divergence comes from the accumulated contribution of arbitrarily short positive durations. Appendix Section A.2 also proves divergence for every dependence structure preserving the distance and duration distributions specified above. The Gaussian fitted to the Figure 4C timing bars also has an infinite inverse-square duration moment.

Every average from finitely many draws is finite with probability one. For independent, identically distributed repetitions of the independent model, the running sample averages tend to infinity with probability one as the number of draws increases, although successive averages can decrease (Appendix Section A.3). A larger simulation therefore cannot establish convergence to a finite unrestricted model mean. The infinite expectation describes the assigned uncertainty model and its extension to arbitrarily short durations.

3 Reproduction results and remaining uncertainty

Reconstructing the bars plotted in Knuth et al.'s Figure 4D gives a mean of $548.5g_0$ and a tallest bin centered at $151.5g_0$. These values agree with their rounded mean and plotted most probable value; their averaging procedure remains unknown. Appendix Section A.1 gives the extraction, calibration, sensitivity bounds and timing-fit methods.

The independent model gives a restricted mean of $595.5g_0$ at an upper limit of $2000g_0$ (Table 1). A separate Gaussian fit to the Figure 4C timing bars, combined with the same angular distribution, gives a restricted mean of $549.9g_0$ and a density mode of $148.1g_0$. The timing fit uses no acceleration summary as an input.

Table 1: Means calculated in this analysis. The Figure 4D bar row spans approximately 20–2000g_0; $a_{\max} = 2000g_0$ marks its plotted edge, and the fraction of Knuth et al.'s samples represented is unknown (n/a). Model rows assume independent angle and duration, condition on $A \le a_{\max}$, and use the angular distribution formed from their stated values.

Basis	$a_{\max}$ (g_0)	Retained probability (%)	Mean (g_0)
Figure 4D bars	2000	n/a	548.5
Independent model	1800	70.8688	563.0
Independent model	2000	72.6361	595.5
Independent model	2200	74.1435	626.0
Figure 4C timing fit	2000	75.5078	549.9

The Figure 4D bar calculation and the model calculations in Table 1 are reproducible from the extraction and propagation definitions in the appendix. The reconstructed $548.5g_0$ is the count-weighted mean of the displayed Figure 4D bins, spanning approximately 20–2000g_0. Knuth et al.'s simulation code, input samples, sample count and averaging rule are unavailable. The plotted bars therefore do not establish which samples entered their reported mean. The $595.5g_0$ independent-model result differs from their reported mean, and the published record does not identify the cause. The timing fit gives a result near the reported value without identifying their averaging procedure. Appendix Section A.1 explains a separate limitation of propagating the plotted distance and timing bars.

4 Scientific implications and reproducible summaries

The distinction changes the scientific interpretation of the reported $550g_0$. Under the independent model, a finite value could describe an average from a particular simulation run or a model mean restricted to a specified range; the unrestricted expectation is infinite. Comparing that value with another reported mean would require matching the averaging definitions. Replication requires the sample size and randomization procedure for a simulation average, or the inclusion range and normalization for a restricted mean. Agreement with the mean reconstructed from the plotted bars leaves Knuth et al.'s averaging rule unresolved.

Under the independent model, f_A and F_A denote the acceleration density and cumulative distribution. The mean restricted to $A \le u$ is

$$\mathbb{E}[A \mid A \le u] = \frac{\int_0^u a f_A(a)\,\mathrm{d}a}{F_A(u)}. \tag{3}$$

The numerator is the retained contribution to the first moment; the denominator is the retained probability. At $u = 2000g_0$, the independent model gives numerator $432.5g_0$ and denominator 0.726361, yielding $595.5g_0$. The restriction excludes 27.4% of probability under this model. The fraction excluded from Knuth et al.'s simulation, if any, is unknown. If the same full sample is retained, changing only the plot's axis limits does not change its average.

Table 1 shows the range sensitivity of the independent model: increasing the upper limit from 1800 to $2200g_0$ changes its conditional mean from 563.0 to $626.0g_0$. These limits illustrate reporting choices. A physically justified restriction would require evidence about the permitted motion or duration.

The unrestricted independent model has a density mode of $173.3g_0$, a median of $704.7g_0$ and an equal-tail 68% interval of $[214.4, 4960.6]g_0$. The interval leaves 16% probability in each tail. These summaries describe uncertainty under the assigned model. Appendix Section A.4 gives the formulas. Separate numerical integrations check these values.

Table 2 in Knuth et al. uses the most probable value $150^{+140}_{-80}g_0$ for the visual departure. Their explicit energy and power calculations concern the separate radar descent in Section 2.4.1 (Knuth et al. 2019). This note supplies no revised value for those results. Divergence of the expectation alone does not invalidate a finite mode. The reconstruction also leaves unexplained the difference between the independent model's $173.3g_0$ density mode and their plotted mode near $150g_0$. Assessment of their broader claims about the encounters requires separate evidence and analysis.

5 Duration uncertainty and simulation practice

Positive-time truncation removes negative durations and retains arbitrarily short positive ones. For the distance distribution defined above, with distance and duration independent, the mean acceleration is finite exactly when the inverse-square duration moment, $\mathbb{E}[T^{-2}]$, is finite. A justified positive minimum duration can ensure this. A lognormal duration or a gamma duration with shape greater than two also has the required finite moment. Each choice needs support from the observation or timing process; obtaining a finite result alone supplies no such support.

The duration mean and variance do not determine whether the acceleration expectation is finite. As a reproducible sensitivity example, Appendix Section A.4 constructs lognormal and gamma durations with exactly the same mean and variance as the positive Gaussian formed from the reported timing values. Their inverse-square moments are 1.5845 and 4.0458 s^{-2}, respectively; the positive Gaussian's corresponding moment is infinite. With the same independent distance distribution, the two finite mean accelerations differ by a factor of approximately 2.55. These alternatives illustrate the effect of the duration law. This comparison provides no observational basis for choosing either replacement.

Knuth et al.'s Bethune analysis uses the same stated Gaussian duration values, 1 ± 1 s, and inverse-square kinematics $A = 4R/T^2$, where R is traveled distance (Knuth et al. 2019, sec. 2.1, eq. (3) and fig. 1). Applying the Gaussian interpretation adopted above, independent positive distance and duration give an infinite expected acceleration. Divergence also holds at any fixed positive distance. The original paper does not fully specify the angular truncation and dependence needed for a complete reconstruction of that case. The approximately $1700g_0$ Bethune summary comes from a logarithmic acceleration histogram; Knuth et al. report no arithmetic mean for this case.

Simulation reports can make the distinction explicit by stating the duration distribution and its truncation, the dependence assumptions, and whether the reported average uses every draw or a selected range. A restricted mean needs its retained probability as well as its bounds. A finite-run mean needs the sampling procedure, number of draws and reproducible randomization settings. A median and probability interval summarize the unrestricted distribution without requiring a finite expectation, although they remain sensitive to the assigned uncertainty model.

6 Conclusion

An average of the displayed bars in Knuth et al.'s Figure 4D reproduces their reported mean near $550g_0$. The independent model has an infinite unrestricted mean acceleration. Interpreting their finite value requires the probability model and averaging procedure actually used, including any sample or range restriction. The independent model gives a reproducible restricted mean of $595.5g_0$ below $2000g_0$. The timing fit to their Figure 4C bars, combined with the same angular distribution and independence assumption, gives $549.9g_0$ under that limit. Their averaging procedure remains unverified.

The analysis specifies what the reported finite mean can represent under the independent model, quantifies the effect of range selection, and identifies a duration-model issue shared with the Bethune kinematics. Duration models with identical means and variances can imply different acceleration expectations. Reproducible averaging rules and an explicit account of short-duration uncertainty are needed to interpret such means. This note supplies no revised numerical value for the modal summary in Table 2 or the separate encounter estimates.

Code and numerical data. The appendix provides the derivations. Numerical code and verification records are available from the author on request.

Appendix. Mathematical derivations

A.1 Histogram reconstruction and timing fit

Two extraction routines read the vector outlines of the bars in Figure 4D of Knuth et al. (2019), recovering 1980 positive-height bars spanning approximately 20–$2000g_0$. The tallest bin is centered at $151.5g_0$. A linear calibration maps PDF horizontal coordinates to acceleration using the ticks labeled 200, 600, 1000, 1400 and $1800g_0$. The outer frame extends beyond the $2000g_0$ tick. The first routine uses PyMuPDF to identify blue, left-to-right horizontal bar-top segments within the panel. In the first routine's coordinates, measured in points from the page's upper-left corner, the acceleration calibration is $a(x)/g_0 = 13.17518660x - 4211.79832129$ and bar height is $695.48803711 - y$. The second routine groups four outline segments per bar and checks the axis calibration against the tick paths. For equal-width bins with lower and upper edges l_j, u_j, centers $c_j = (l_j + u_j)/2$ and heights h_j proportional to counts, the bar-count mean is

$$\bar{A}_{\text{bins}} = \frac{\sum_j h_j c_j}{\sum_j h_j}. \tag{A.1}$$

The 1980 positive-height bars give $548.515g_0$; weighting by bar area to check small coordinate-width differences gives $548.534g_0$. With fixed edges and heights, any sample average consistent with the bins lies between $\sum_j h_j l_j / \sum_j h_j$ and $\sum_j h_j u_j / \sum_j h_j$, or approximately 548.016 and $549.015g_0$. A coordinate sensitivity calculation allowing 0.001 PDF point per coordinate, combined with the within-bin allowance, gives approximately $[547.94, 549.10]g_0$. This reconstruction range describes coordinate and bin-position uncertainty. It supplies no statistical confidence interval and does not cover axis-label errors or unplotted samples.

The timing reconstruction calibrates Knuth et al.'s Figure 4C from the ticks labeled 0 to 5 s. An unweighted least-squares fit to the extracted positive bar heights uses

$$h(t) = H \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right], \qquad \mu = 1.10456980 \text{ s}, \quad \sigma = 1.07252203 \text{ s}, \tag{A.2}$$

with free amplitude H. The fit uses the timing panel alone; no acceleration mean enters the fit. Combining this fitted positive-time Gaussian with the independent model's angular distribution, while assuming angle-duration independence, gives $\mathbb{E}[A \mid A \le 2000g_0] = 549.901g_0$. The fitted parameters describe the plotted coordinates. Knuth et al.'s simulation settings, sample inclusion, and averaging procedure remain unknown.

With the same angular distribution and independence assumption, conditioning on $20g_0 \le A \le 2000g_0$ rather than $A \le 2000g_0$ changes the conditional mean by less than $0.01g_0$ under either the positive-time Gaussian of Section 2 or the law fitted to Figure 4C. Treating the plotted distance and timing histograms as independent marginals gives $561.4g_0$ below $2000g_0$. The distance panel starts at 2 mi and excludes the low-distance portion of the angularly derived distance distribution used in the independent model. The plotted marginal histograms also leave their original pairing of distance and timing samples unknown. These limitations prevent identification of Knuth et al.'s complete computation from the plotted panels.

A.2 Divergence for every coupling

Knuth et al. use $A = 2D/T^2$ and $D = (w/2)\cot(\theta/2)$, with their stated $w = 3.048$ m. This note treats their 2 ± 1 arcmin angular values as Gaussian location and scale before conditioning on $\theta_0 \le \theta < \pi$, where $\theta_0 = 1$ arcmin. Angles in formulas are expressed in radians. Knuth et al. specify the lower cutoff; the added upper restriction keeps $D > 0$. For duration, this note likewise treats Knuth et al.'s 1 ± 1 s values as Gaussian location and scale before truncation to $T > 0$, giving density

$$f_T(t) = \frac{\phi((t-\mu)/\sigma)}{\sigma\Phi(\mu/\sigma)}, \quad t > 0, \qquad \mu = \sigma = 1 \text{ s}, \tag{A.3}$$

where ϕ and Φ are the standard normal density and distribution function. For either timing law, F_T denotes its cumulative distribution. The timing law fitted to Figure 4C in Section A.1 also has $f_T(0+) > 0$. The following proof applies to either timing law.

$g(\theta)$ denotes the bounded, normalized angular density, and $G = \sup g$. The distance lies in $(0, d_{\max}]$, where $d_{\max} = (w/2)\cot(\theta_0/2)$. Inverting and differentiating gives

$$\theta(d) = 2\arctan\frac{w}{2d}, \qquad f_D(d) = g(\theta(d))\frac{4w}{w^2+4d^2} \le M := \frac{4G}{w}. \tag{A.4}$$

With F_D denoting the distance cumulative distribution, the density bound implies $F_D(d) \le Md$ for $d \ge 0$. The positive limit $f_T(0+) > 0$ implies that there are $b, \delta > 0$ with $F_T(t) \ge bt$ for $0 < t < \delta$.

A coupling is any joint distribution with the angularly derived distance and positive-time duration marginals defined above. For every coupling and $a, t > 0$, $\{T \le t, D > at^2/2\} \subseteq \{A > a\}$. For $0 < t < \delta$, the union bound gives

$$\Pr(A > a) \ge F_T(t) - F_D(at^2/2) \ge bt - \frac{Mat^2}{2}. \tag{A.5}$$

For $a > b/(M\delta)$, choosing $t = b/(Ma) < \delta$ yields

$$\Pr(A > a) \ge \frac{b^2}{2Ma}, \qquad \mathbb{E}[A] = \int_0^\infty \Pr(A > a)\,\mathrm{d}a = \infty. \tag{A.6}$$

The expectation identity follows by integrating $A = \int_0^\infty \mathbf{1}_{\{A>a\}}\,\mathrm{d}a$ and exchanging nonnegative integrals. The bound proves divergence for the distance and duration marginals defined above without determining the exact tail probability.

A.3 Finite simulation averages

Each simulated acceleration is finite almost surely because D is bounded and $T > 0$. Every finite sample average is therefore finite almost surely. For independent, identically distributed repetitions $A_1, A_2, \ldots$, the capped values are $A_i^{(K)} = \min(A_i, K)$ at acceleration threshold $K > 0$. The strong law for these bounded variables gives

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} A_i \geq \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} A_i^{(K)} = \mathbb{E}[\min(A, K)] \quad \text{almost surely.} \tag{A.7}$$

This holds simultaneously for the countable thresholds $K = jg_0$, $j = 1, 2, \ldots$. As $j \to \infty$, monotone convergence gives $\mathbb{E}[\min(A, jg_0)] \to \mathbb{E}[A] = \infty$. The sample averages tend to infinity almost surely; individual successive averages can decrease. The result requires independence across repetitions, allows dependence within each pair (D, T), and supplies no finite sample size at which a particular average must be exceeded.

A.4 Restricted means and finite probability summaries

The calculations below assume independence between the angularly derived distance and either positive-time duration law introduced above, before conditioning on acceleration. The duration survival function is $S_T = 1 - F_T$. For fixed $D = d$ and $a > 0$, the event $A \leq a$ is equivalent to $T \geq \sqrt{2d/a}$. The notation $\mathbb{E}_D$ means expectation over the distance distribution. Averaging over distance gives F_A; differentiating gives f_A:

$$F_A(a) = \mathbb{E}_D\left[S_T\left(\sqrt{2D/a}\right)\right], \qquad f_A(a) = \mathbb{E}_D\left[\frac{\sqrt{2D}}{2a^{3/2}} f_T\left(\sqrt{2D/a}\right)\right]. \tag{A.8}$$

The chain rule uses $\partial\sqrt{2d/a}/\partial a = -\sqrt{2d/a}/(2a)$ and $S_T' = -f_T$. Bounded D and bounded f_T justify differentiation on compact intervals with $a > 0$. Quantiles solve $F_A(a_q) = q$ for $0 < q < 1$; threshold probabilities are $\Pr(A > a) = 1 - F_A(a)$. These summaries remain finite without a finite mean.

For a finite upper acceleration $u > 0$, the retained probability is $Q(u) = F_A(u)$ and the partial first moment is $R(u) = \mathbb{E}[A\mathbf{1}_{\{A\leq u\}}]$. Since $\min(A, u) = A\mathbf{1}_{\{A\leq u\}} + u\mathbf{1}_{\{A>u\}}$, its survival integral gives

$$R(u) = \int_0^u [1 - F_A(a)]\,\mathrm{d}a - u[1 - F_A(u)] = \int_0^u a f_A(a)\,\mathrm{d}a, \tag{A.9}$$

$$\mathbb{E}[A \mid A \leq u] = \frac{R(u)}{Q(u)}. \tag{A.10}$$

Both integrals are finite because the retained acceleration is bounded. Independent propagation also gives the equivalent expression

$$R(u) = \mathbb{E}_D\left[2D\int_{\sqrt{2D/u}}^{\infty} \frac{f_T(t)}{t^2}\,\mathrm{d}t\right]. \tag{A.11}$$

For $0 \leq l < u$ with $F_A(u) > F_A(l)$, the mean over that range is

$$\mathbb{E}[A \mid l \leq A \leq u] = \frac{R(u) - R(l)}{F_A(u) - F_A(l)}, \qquad R(0) = F_A(0) = 0. \tag{A.12}$$

The denominator is the retained probability. Restricting a plot's axis does not define this conditional mean. Averaging the Figure 4D bars performs a normalization over their represented counts. The published description does not establish whether their reported mean used those counts.

The model-based means were evaluated by adaptive integration over angle and duration. A 512-node angular quadrature and the survival-integral expression for $R(u)$ provide separate numerical checks. The adaptive calculation integrates angles from 1 to 16 arcmin, omitting normalized angular probability below 10^{-44}. The separate quadrature extends to 14 arcmin, omitting probability below 2.2×10^{-33}. The conditional means agree to the displayed precision.

Duration models with matched moments. For independent D, T with $0 < \mathbb{E}[D] < \infty$, the identity $\mathbb{E}[A] = 2\mathbb{E}[D]\mathbb{E}[T^{-2}]$ shows that the mean is finite exactly when $\mathbb{E}[T^{-2}] < \infty$. For any $\delta > 0$, the contribution to $\mathbb{E}[T^{-2}]$ from $t \geq \delta$ is at most δ^{-2}, so finiteness depends on $\int_0^\delta f_T(t) t^{-2}\,\mathrm{d}t$. In particular, if $f_T(t) \sim ct^\alpha$ as $t \downarrow 0$ with $c > 0$, the mean is finite exactly when $\alpha > 1$. A positive minimum duration guarantees a finite inverse-square moment. With dependence, the corresponding integrand includes $\mathbb{E}[D \mid T = t] f_T(t)/t^2$ and requires a separate assessment.

The following example changes only the duration law and preserves its mean and variance. The dimensionless duration is $X = T/(1\,\mathrm{s})$. Under the Gaussian interpretation adopted above, the reported 1 ± 1 s timing values specify the location and scale before conditioning on $X > 0$. For that distribution, λ, the mean and the variance are

$$\lambda = \frac{\phi(1)}{\Phi(1)}, \qquad m = \mathbb{E}[X] = 1 + \lambda, \qquad v = \mathrm{Var}(X) = 1 - \lambda - \lambda^2. \tag{A.13}$$

Numerically, $m = 1.28759997094$ and $v = 0.629686285777$. These are the mean and variance after truncation; the underlying Gaussian location and scale are both 1.

Two alternative distributions for X with this same m, v are:

$$X_\mathrm{G} \sim \mathrm{Gamma}(k, \beta), \qquad k = \frac{m^2}{v}, \qquad \beta = \frac{m}{v}, \tag{A.14}$$

$$\log X_\mathrm{L} \sim N(\eta, s^2), \qquad s^2 = \log(1 + v/m^2), \qquad \eta = \log m - \frac{s^2}{2}. \tag{A.15}$$

The gamma density is $\beta^k x^{k-1} e^{-\beta x}/\Gamma(k)$ for $x > 0$; β is a rate. Its mean and variance are $k/\beta = m$ and $k/\beta^2 = v$. The lognormal mean and variance are $e^{\eta + s^2/2} = m$ and $(e^{s^2} - 1)e^{2\eta + s^2} = v$. The parameters are $k = 2.63292011$, $\beta = 2.04482772$, $\eta = 0.09180839$ and $s^2 = 0.32194322$.

Integrating the gamma density after multiplication by x^{-2} and using $\Gamma(k) = (k-1)(k-2)\Gamma(k-2)$ gives the first identity below. Completing the square in the normal density gives the second:

$$\mathbb{E}[X_\mathrm{G}^{-2}] = \frac{\beta^2}{(k-1)(k-2)} = 4.04575532, \tag{A.16}$$

$$\mathbb{E}[X_\mathrm{L}^{-2}] = e^{-2\eta + 2s^2} = 1.58450123. \tag{A.17}$$

The positive-time Gaussian formed from the stated timing values has $\mathbb{E}[X^{-2}] = \infty$. Multiplication by $(1\,\mathrm{s})^{-2}$ gives the corresponding duration moments. For the same independent distance distribution, each mean acceleration equals $2\mathbb{E}[D]\mathbb{E}[T^{-2}]$. The two finite expectations differ by a factor of approximately 2.55, despite equal duration means and variances. This example supplies no empirical preference between the duration laws.

For the Bethune relation $A = 4R/T^2$, any fixed $R = r > 0$ and the positive-time Gaussian formed from the stated timing values give $\mathbb{E}[A] = 4r\mathbb{E}[T^{-2}] = \infty$. Under independence, random positive R also gives divergence: some $r_0 > 0$ has $\Pr(R \geq r_0) > 0$, and $\mathbb{E}[A] \geq 4r_0 \Pr(R \geq r_0)\mathbb{E}[T^{-2}] = \infty$. This argument uses the stated duration interpretation without reconstructing the Bethune distance distribution or asserting that its mean is finite.

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