Common E-Values Across Deterministic Reporting Protocols

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Abstract

A deterministic reporting protocol partitions an outcome space into reported events. Requiring identical events to receive identical multipliers couples calibration across protocols. For a compact convex positive null class and positive alternative on a finite outcome space, a pair admits a common calibrated score with positive log growth in every model making both protocols informative exactly when the partitions are comparable by refinement or have no identical cell. Every remaining pair admits an obstructing positive null segment. Pairwise compatibility does not suffice for families. Two sufficient classes and a necessary condition leave two structural questions open. On at most three outcomes, one positive rule is valid under every deterministic partition and grows under each informative one; a stable four-outcome example attains the minimum for failure. Finitely many reports on a countable outcome space reduce exactly to a finite model. For randomized coarsening-at-random channels, equal scores on equal truthful subset labels admit a common growing rule on at most two outcomes, while two informative channels can be obstructed on three. Under pre-outcome joint selection of null laws and deterministic protocols, disjoint report hulls characterize existence of an event-history e-process with a uniform positive exponential growth rate under every alternative protocol policy. Pooled history scores grow under alternative protocol changes and remain valid under each fixed null law. The four-outcome certificate gives a changing-null policy under which the pooled score diverges and violates stopped and deterministic-time validity.

Keywords: e-values; reporting protocols; composite hypotheses; log growth; statistical evidence.

1 The compatibility question

An investigator or an account of statistical evidence can impose an event-identity requirement: the same reported event must receive the same multiplier under every retained reporting protocol. Chugg et al. (2026) record an evidential desideratum: a measure of evidence should depend only on the realized event and not on unrealized contingencies of the experimental protocol. The requirement applies that desideratum to one-step multiplicative scores and makes the multiplier invariant under a change of protocol that leaves the reported event unchanged, even when the protocol is recorded. I analyze whether this requirement can coexist with null calibration and positive growth under each informative protocol. The analysis identifies exactly which pairs of protocols meet all three demands in every informative model, and it shows that the answer for larger families depends on more than their pairwise structure. It does not establish that an actual statistical workflow needs the requirement.

Missing information can also force the requirement. When an analyst observes the reported event but not the protocol that produced it, a score can use only the event, and it must be valid under each candidate protocol. Chugg et al. (2026) describe binary data $(0, 1, 0, 1, 1)$ whose sampling plan is unknown: the plan may have fixed the sample size at five or stopped at the first two consecutive

ones. They note that an analyst often learns which event occurred without knowing how the reporting variable was defined, and they list isolated e-variables among the measures that fail the desideratum above. Cap the stopping plan at N observations, so that both plans partition $\{0,1\}^N$. For $N = 5$, the fixed plan refines the stopping plan. For every $N \geq 6$, neither plan refines the other: the stopping cell of sequences beginning 11 contains several fixed-plan cells, and the fixed-plan cell of sequences beginning 00000 contains several stopping cells. The two plans share exactly three cells: the sets of sequences whose first five entries are 00011, 01011, or 10011. A fixed-plan cell has 2^{N-5} elements, and a stopping cell that ends at observation t has 2^{N-t}, so a shared cell must stop at observation five; its first two consecutive ones then occupy positions four and five, so position three holds a zero and the first two entries are not both ones, which leaves exactly these three prefixes. The observed record determines the second of these cells. For each such N, Theorem 7 therefore gives a positive model that makes both capped plans informative while no common valid score grows under both. This conclusion concerns composite nulls. With a simple null and a simple alternative, the event likelihood ratio is valid under both plans and grows under each informative one (Section 5). The obstructing model of Theorem 7 need not be an independent Bernoulli model, so the conclusion does not show that a composite Bernoulli null obstructs this pair. The voltmeter example of Chugg et al. (2026), discretized to finitely many readings, falls in the refinement branch of Theorem 7: censoring merges the readings at or above the censoring point and keeps the others, so the uncensored protocol refines the censored one. These examples come from the literature on evidential desiderata and do not document an applied workflow.

Consider a hypothetical laboratory with four ordered reading bins. A technician selects a setting before taking a reading. One setting reports bin 1, bins 2 and 3 together, or bin 4. The other reports bins 1 and 2 together, bin 3, or bin 4. The investigator requires one multiplier for each reported interval, and an analyst updates an accumulated score using that entry. The technician can record the setting separately, but the table uses only the interval: a bin 4 report must select the same entry under either setting. This rule permits an audit to determine the multiplier from the reported interval alone and imposes an equality constraint on the entries. Section 4.1 gives the probabilities for this constructed example and checks whether one table can support growth under both settings.

A registered analysis plan can impose this restriction before the reporting setting is known. In a hypothetical multi-site study, the plan can assign one multiplier to each reportable interval while each site's instrument determines its reporting resolution. An audit rule can then require identical intervals to select the same multiplier, using the reported interval as the complete scoring input. The classification determines which menus of resolutions can satisfy the common-table constraint in every informative model; it is exact for pairs and partial for larger menus. Section 7 shows when retaining history permits growth and when conditional null changes preserve the obstruction. This hypothetical plan supplies no evidence of practical demand for the requirement.

An e-variable is a nonnegative statistic with expectation at most one under each null distribution. Its realized value is an e-value. Conditional versions of this expectation bound justify multiplication through time. Expected log growth under a specified alternative measures the asymptotic performance of fixed repeated multiplication. E-value calibration and log-growth criteria are established parts of safe testing and information-projection theory (Grünwald et al., 2024; Larsson et al., 2025). They do not determine whether the same reported event must receive the same score in different experiments.

Separate calibration does not answer the common-table question. An informative protocol can distinguish the alternative from every projected null and therefore admits its own score with positive log growth. Its score for a shared report may still be incompatible with the entries required by another protocol. I ask only for some positive growth under each protocol, even at a cost in

its separate performance. When this weaker demand fails, no common table grows under both protocols, whatever separate performance it gives up.

The main result identifies exactly which pairs avoid this failure for every informative positive model. Either one partition refines the other, so a score on the coarser reports can be reused, or the two partitions have no identical cell, so their entries can be chosen separately. Every other pair admits a positive null segment that prevents common growth. This is a classification of reporting structures, with a model-dependent score allowed in each case. A separate construction shows that on at most three outcomes one table works across all informative deterministic partitions at once. The four-bin example attains the first possible failure for deterministic partitions.

The core structural results are the pair classification in Theorem 7, the simultaneous construction on at most three outcomes in Theorem 13, and the family counterexample in Proposition 8. Lemma 10 and Corollary 11 give a necessary condition for families, Corollary 14 gives a second sufficient class on every outcome count, and Section 5.1 states two open questions. The report-space test applies established separation and dilution methods after identical reports have been identified as one coordinate. The explicit certificate quantifies a stable deterministic obstruction, the component condition supplies a sufficient design rule, and the countable result is a finite-report reduction. Theorem 5 marks a boundary of the deterministic restriction: common scores on truthful subset labels can fail on three outcomes for randomized coarsening-at-random channels, and two outcomes always suffice. The sequential analysis extends the report-space criterion to arbitrary event-history scores under conditional null changes and gives an explicit pooled-score failure.

2 Model and scoring requirements

Let $\Omega = \{1, \dots, m\}$. Let $\mathcal{C}$ be a nonempty compact convex subset of the strictly positive probability simplex, and let Q be a strictly positive probability distribution. Compactness supplies a common positive lower bound for $P(i)$ over all $P \in \mathcal{C}$ and $i \in \Omega$. The outcome space, null class, and alternative remain fixed within each comparison.

A deterministic protocol is a partition Π of Ω into nonempty cells. When the outcome is i, the report is the unique cell $A \in \Pi$ containing i. The protocol is chosen before the current outcome. A score e_Π is valid if

$$e_\Pi(A) \geq 0, \qquad \sum_{A \in \Pi} P(A) e_\Pi(A) \leq 1 \quad \text{for every } P \in \mathcal{C}. \tag{1}$$

Markov's inequality gives $P(e_\Pi \geq 1/\alpha) \leq \alpha$ for $0 < \alpha < 1$. This bound does not assign a posterior probability to the null.

Write $T_\Pi P = (P(A))_{A \in \Pi}$, $\mathcal{C}_\Pi = T_\Pi \mathcal{C}$, and $Q_\Pi = T_\Pi Q$. A protocol is *informative* when $Q_\Pi \notin \mathcal{C}_\Pi$. This is distinguishability relative to the stated model. Its expected log score is

$$G_\Pi(e) = \sum_{A \in \Pi} Q(A) \log e(A). \tag{2}$$

All logarithms are natural. A zero score at any possible report gives growth $-\infty$. For a fixed positive score and independent repetitions from Q, the strong law gives convergence of the average log product to (2).

For example, suppose a single null assigns probability $1/2$ to each of two reports. Scores $3/2$ and $1/2$ are valid because their null mean is one. If the alternative assigns probabilities $3/4$ and $1/4$, their expected log score is $\frac{1}{4}\log(27/16) > 0$. Under independent repetitions from this alternative, the accumulated score has positive asymptotic log growth. The constant score one is also valid, but

its log growth is zero. Calibration controls the mean under the null; growth evaluates repeated use under the alternative.

Before coupling protocols, I need to know what each can achieve separately. The following established projection formula supplies a valid positive-growth score whenever the reported probabilities distinguish the alternative from the null class. It also supplies the score that the refinement construction will reuse. For positive probability vectors, put $D(q\|p) = \sum_i q_i \log(q_i/p_i)$.

Lemma 1 (Reverse information projection on one protocol)**.** *Let p^*_Π minimize $D(Q_\Pi\|p)$ over $\mathcal{C}_\Pi$. The minimizing report distribution is unique. The unique log-optimal valid score is*

$$e^*_\Pi(A) = \frac{Q(A)}{p^*_\Pi(A)}, \qquad G^*_\Pi = D(Q_\Pi\|p^*_\Pi).$$

Its growth is positive exactly when the protocol is informative.

Proof. Compactness and positivity give existence. Strict convexity of the divergence in p gives uniqueness. Its directional derivative at the minimum yields

$$\sum_{A\in\Pi} \frac{Q(A)}{p^*_\Pi(A)} p(A) \le 1 \quad (p \in \mathcal{C}_\Pi),$$

which proves validity. For any positive valid e, Jensen's inequality gives

$$\sum_A Q(A) \log \frac{e(A)}{e^*_\Pi(A)} \le \log \sum_A p^*_\Pi(A) e(A) \le 0.$$

Strict concavity of log gives uniqueness of the optimal score. The relative entropy vanishes exactly when the two distributions agree. □

This is the finite positive case of reverse information projection (Grünwald et al., 2024; Larsson et al., 2025). Several master distributions can produce the unique minimizing report distribution.

Let $\mathcal{V}$ be a nonempty collection of protocols requiring validity, and let $\varnothing \ne \mathcal{H} \subseteq \mathcal{V}$ be those requiring positive growth. Both sets are finite. Define the common report alphabet

$$Y = \{A : A \in \Pi \text{ for some } \Pi \in \mathcal{V}\}.$$

Definition 2 (Common event rule)**.** A common rule is a function $F : Y \to [0, \infty)$. It is $\mathcal{V}$-valid if

$$\sum_{A\in\Pi} P(A)F(A) \le 1 \quad (P \in \mathcal{C},\ \Pi \in \mathcal{V}). \tag{3}$$

It grows across $\mathcal{H}$ if $G_\Pi(F) > 0$ for each $\Pi \in \mathcal{H}$.

The investigator can know the protocol while requiring its score to depend only on the event. The equality constraint is a restriction on the numerical rule. It does not model forgetfulness. It is also weaker than a full likelihood-based equivalence condition: different events can have proportional likelihoods without receiving equal scores under this definition. Nor does it require one master statistic measurable under every partition. Distinct reports containing the same outcome can have different scores; only identical reports share a coordinate.

This distinction explains why merging separately calibrated scores does not by itself solve the problem. E-value merging combines numerical scores defined on one observation space (Vovk and Wang, 2021). A merger can vary between underlying outcomes hidden by a single report, so it need

not define a function of that report. Here the output must be one number for each event in Y, regardless of which retained protocol reports it.

The event requirement is motivated by the extension question raised by Chugg et al. (2026): a random-variable value identifies an event, while e-process validity refers to a filtration. Restricting attention to finite deterministic reports makes one part of that question precise: whether the same event score can serve as a calibrated multiplicative component under several protocols. This restriction leaves arbitrary event extensions of e-processes and general evidential equivalence outside the result. Section 7 returns to event identity for entire histories.

The cited literature supplies the tools used here. Reverse information projection gives the one-protocol log-optimal score (Grünwald et al., 2024; Larsson et al., 2025). Conditional calibration justifies multiplication through time (Grünwald et al., 2024). Convex separation and dilution give the report-space test (Zhang et al., 2024). The likelihood envelope gives the pooled history score (Koolen and Grünwald, 2022). Ramdas et al. (2022) distinguish e-processes from nonnegative supermartingales and analyze validity under changes of the conditional null law. E-value merging combines scores on one observation space (Vovk and Wang, 2021) and does not force equal entries on identical reports. Filtration lifting transports a powerful e-process to a finer filtration (Choe and Ramdas, 2026). The refinement branch of Theorem 7 is its one-step static counterpart: the lifted score is the same function of the outcome and needs no adjuster. Section 4.1 distinguishes the model from robust updating (van Ommen et al., 2016); Section 4.2 uses the established coarsening-at-random geometry of Gill and Grünwald (2008). The structural contributions are the pair classification with its converse on the original outcome space, the family results, the sharp deterministic outcome count, and the three-outcome CAR obstruction with its two-outcome lower bound. Theorem 18 gives a finite reporting consequence of established convexification and null-switching arguments, with the witness realized on the master space. Proposition 17 uses a standard count-martingale argument. No priority is claimed for that argument, the CAR mechanisms, or a general convexification principle, and no exhaustive priority claim is made.

3 A report-space test for common growth

Embed each report distribution into Y by

$$P^{[\Pi]}(A) = P(A)\mathbf{1}_{\{A\in\Pi\}}, \qquad Q^{[\Pi]}(A) = Q(A)\mathbf{1}_{\{A\in\Pi\}}.$$

Zeros record reports unavailable under that protocol. Define

$$N_{\mathcal{V}} = \operatorname{conv}\{P^{[\Pi]} : P \in \mathcal{C}, \Pi \in \mathcal{V}\}, \qquad A_{\mathcal{H}} = \operatorname{conv}\{Q^{[\Pi]} : \Pi \in \mathcal{H}\}. \tag{4}$$

These are compact convex sets of probabilities on the same finite alphabet. A common rule is exactly one e-variable for $N_{\mathcal{V}}$.

Under a common table, the mean score depends only on the law of the reported event on the common alphabet; it does not depend on which protocols produced that law. If a mixture of null report laws equals a mixture of alternative report laws, calibration therefore bounds the alternative mixture's mean by one. At least one alternative report law must then have mean score at most one, which excludes positive expected log growth there. The mixtures express simultaneous constraints on the table; they do not specify how a reporter chooses a protocol.

Proposition 3 (Common growth and separation on the report alphabet)**.** *There exists a strictly positive $\mathcal{V}$-valid common rule that grows across $\mathcal{H}$ if and only if $N_{\mathcal{V}} \cap A_{\mathcal{H}} = \varnothing$. When the two sets intersect, no $\mathcal{V}$-valid common rule, positive or not, grows across $\mathcal{H}$. The same condition is equivalent*

to existence of a common valid rule whose expected score exceeds one under every performance protocol.

Proof. If $r = \sum_{\Pi \in \mathcal{H}} \lambda_\Pi Q^{[\Pi]}$ belongs to both sets, validity gives $\mathbb{E}_r F \le 1$, so the corresponding alternative expected scores cannot all exceed one. If F vanishes on any report of a performance protocol, that protocol has growth $-\infty$ and simultaneous positive growth already fails. Otherwise every performance growth is finite, and Jensen's inequality gives

$$\sum_{\Pi \in \mathcal{H}} \lambda_\Pi G_\Pi(F) = \mathbb{E}_r \log F \le \log \mathbb{E}_r F \le 0.$$

If the compact convex sets are disjoint, strict separation supplies a vector h and $\delta > 0$ such that

$$\sup_{n \in N_\mathcal{V}} n \cdot h = 0, \qquad \inf_{a \in A_\mathcal{H}} a \cdot h \ge \delta.$$

Subtracting the null supremum from any strictly separating vector gives this normalization because all vectors have total mass one. For sufficiently small $\varepsilon > 0$, the rule $F = 1 + \varepsilon h$ is positive and valid. Each growth function has derivative at zero at least δ. A common sufficiently small ε gives positive growth in the finite family $\mathcal{H}$. Its expected score also exceeds one there. □

The proposition applies two established results (Zhang et al., 2024): an e-variable whose expectation exceeds one under every alternative law exists exactly when the convex hulls of the null and alternative families are disjoint, and a bounded e-variable of that kind converts into one with positive expected logarithm. The same source states a fuller equivalence, which adds existence of a nontrivial p-variable and of a bounded e-variable with positive expected logarithm under every alternative law, under a joint atomlessness assumption. The proof here uses a finite report alphabet, finite-dimensional separation, and a bounded dilution around one, so it requires no atomless observation space or auxiliary atomless randomization. Its role is to place every protocol on the common report alphabet, where identical events share one coordinate. Checking $Q_\Pi \notin \mathcal{C}_\Pi$ separately cannot detect a null mixture from several protocols that agrees with an alternative mixture on that alphabet.

For a polytope null, list its vertex report laws as columns of N, and list the performance report laws as columns of M. An obstruction is certified by

$$Nw = Mv, \qquad w, v \ge 0, \qquad \mathbf{1}^T w = \mathbf{1}^T v = 1. \tag{5}$$

This is a finite linear feasibility problem. The mixtures certify simultaneous inequalities; they do not posit a hypothesis-dependent reporting mechanism. Enumerating all partitions is unnecessary when the reporting domain is given explicitly, and no efficiency claim is made for enumeration over an unrestricted domain.

4 Two informative protocols with one shared event

A single shared entry can couple otherwise separate score tables strongly enough to prevent growth. I construct a model in which each protocol distinguishes Q from the null segment, but their report laws have the mixture equality excluded by Proposition 3. The labels below simplify the certificate; exchanging labels 2 and 3 gives the ordered-bin example from the introduction.

Take

$$Q = \tfrac{1}{4}(1,1,1,1), \quad R_0 = \tfrac{1}{60}(6,21,27,6), \quad R_1 = \tfrac{1}{60}(16,3,17,24), \quad \mathcal{C} = \operatorname{conv}\{R_0, R_1\}. \tag{6}$$

Retain exactly

$$\Sigma_1 = \{\{1\},\{2,3\},\{4\}\}, \qquad \Sigma_2 = \{\{2\},\{1,3\},\{4\}\}, \qquad \mathcal{V} = \mathcal{H} = \{\Sigma_1, \Sigma_2\}. \tag{7}$$

Only $\{4\}$ is shared.

Both protocols are informative. The Σ_1-measurable function $\phi = (1, 0, 0, -5/9)$ has expectation $2/45$ under both null endpoints and $1/9$ under Q. Under Σ_2, the event $\{1,3\}$ has null probability $11/20$ throughout $\mathcal{C}$ and alternative probability $1/2$.

Table 1 displays the report probabilities that enter the common-table constraints. The shared event has one column; a zero means that the protocol cannot issue that report. Each row separately evaluates the same five score entries. The proof exhibits a mixture of the first two rows equal to a mixture of the last four, which is the joint obstruction that separate informativeness does not detect.

Table 1: Report laws on the common five-event alphabet.

Law	$\{1\}$	$\{2,3\}$	$\{2\}$	$\{1,3\}$	$\{4\}$
$Q^{[1]}$	$1/4$	$1/2$	0	0	$1/4$
$Q^{[2]}$	0	0	$1/4$	$1/2$	$1/4$
$R_0^{[1]}$	$1/10$	$4/5$	0	0	$1/10$
$R_1^{[1]}$	$4/15$	$1/3$	0	0	$2/5$
$R_0^{[2]}$	0	0	$7/20$	$11/20$	$1/10$
$R_1^{[2]}$	0	0	$1/20$	$11/20$	$2/5$

Theorem 4 (A two-protocol obstruction)**.** *Every common valid rule for* (6)–(7) *satisfies*

$$\tfrac{9}{20} G_{\Sigma_1}(F) + \tfrac{11}{20} G_{\Sigma_2}(F) \le 0. \tag{8}$$

Equality holds exactly when all five report scores equal one. Every other common valid rule has negative growth under at least one informative protocol. These conclusions persist if each coordinate of Q, R_0, and R_1 changes by at most $3/5000$, subject to remaining probability vectors, with continuously varying mixture weights replacing $9/20$ and the null weights $(1/8, 3/8, 3/8, 1/8)$.

Proof. Use the alphabet $Y = (\{1\},\{2,3\},\{2\},\{1,3\},\{4\})$. Direct calculation gives

$$\begin{aligned} r &= \tfrac{9}{20} Q^{[1]} + \tfrac{11}{20} Q^{[2]} \\ &= \tfrac{1}{8} R_0^{[1]} + \tfrac{3}{8} R_1^{[1]} + \tfrac{3}{8} R_0^{[2]} + \tfrac{1}{8} R_1^{[2]} \\ &= (9/80, 9/40, 11/80, 11/40, 1/4). \end{aligned} \tag{9}$$

Validity gives $\mathbb{E}_r F \le 1$. Jensen's inequality proves (8). Since r is positive on all five reports, equality requires a constant score, and equality at zero fixes that constant at one. A zero score gives $-\infty$ under at least one protocol.

For the neighborhood claim, let $a = Q^{[1]}$, $b = Q^{[2]}$, and form

$$N = \begin{pmatrix} 1/10 & 4/15 & 0 & 0 \\ 4/5 & 1/3 & 0 & 0 \\ 0 & 0 & 7/20 & 1/20 \\ 0 & 0 & 11/20 & 11/20 \\ 1/10 & 2/5 & 1/10 & 2/5 \end{pmatrix}.$$

The columns are the four null report laws. Put $M = [N, -(a-b)]$ and $x = (1/8, 3/8, 3/8, 1/8, 9/20)^T$. Direct inversion gives $\det M = -3/500$, $Mx = b$, and $\|M^{-1}\|_\infty = 111/5$, where the matrix norm is the maximum absolute row sum. If each master probability changes by at most η, each null report entry changes by at most 2η, each entry of $a - b$ by at most 4η, and each entry of b by at most 2η. These bounds give

$$\|\Delta M\|_\infty \le 12\eta, \qquad \|\Delta b\|_\infty \le 2\eta.$$

For $\eta \le 3/5000$, $\|M^{-1}\Delta M\|_\infty < 1$. The identity

$$\Delta x = (I + M^{-1}\Delta M)^{-1} M^{-1}(\Delta b - \Delta M x)$$

and the geometric-series bound on the inverse yield

$$\|\Delta x\|_\infty \le \frac{(111/5)(2\eta + 12\eta \cdot 9/20)}{1 - (111/5)12\eta} \le \frac{12321}{105020} < \frac{1}{8}.$$

All four null weights stay positive, and the alternative weight stays in $(0, 1)$. Summing the perturbed report coordinates forces the null weights to sum to one. The smallest original master probability is $1/20$, so all master laws remain positive. The separating gap for ϕ is at least $1/15 - (28/9)\eta > 0$, since each of its two expectations changes by at most $(14/9)\eta$. The $\{1, 3\}$ probability gap is at least $1/20 - 4\eta > 0$. Both protocols remain informative. The perturbed certificate has positive mass on every report, so the same Jensen and equality arguments apply. □

The certificate also prevents both alternative expected scores from exceeding one. For any increasing concave utility u with $u(1) = 0$, it gives the same nonpositive weighted expected utility whenever defined. Logarithms have the further operational role of describing products.

Removing either protocol permits its separate positive-growth optimum. Allowing scores to depend on (Π, A) also permits both separate optima. These changes remove a stated joint requirement. The theorem does not require calibration under an unused full-report protocol, and it does not fix a reference score.

4.1 An interface for ordered measurements

Consider four ordered measurement bins $1 < 2 < 3 < 4$. An instrument has two preselected settings, each restricted to three consecutive interval reports:

$$\Pi_L = \{\{1\}, \{2, 3\}, \{4\}\}, \qquad \Pi_R = \{\{1, 2\}, \{3\}, \{4\}\}.$$

The settings move one lower threshold and preserve the upper-bin report. Take a uniform alternative and the null segment with endpoints

$$\widetilde{R}_0 = \tfrac{1}{60}(6, 27, 21, 6), \qquad \widetilde{R}_1 = \tfrac{1}{60}(16, 17, 3, 24).$$

The segment represents uncertainty about a mixing weight between two specified null operating regimes. Swapping labels 2 and 3 in (6)–(7) gives exactly this model and these partitions. Both settings are informative, and Theorem 4 retains the weights $9/20$ and $11/20$. Consecutive bins, equal report counts, and a stable upper report therefore do not suffice for a common growing score table.

Suppose the accumulator accepts the current interval as its only scoring input and retains only its current wealth. Its update uses one predeclared factor $F(A)$, so the unchanged upper report must use one entry under both settings. Requiring conditional calibration makes that entry reusable

when the setting or null mixing weight changes before the next measurement. If the design also requires positive growth under repeated use of either informative setting, the displayed model makes those requirements inconsistent.

The finest partition coarser than both settings is $\Gamma = \{\{1,2,3\},\{4\}\}$. Every common coarsening is Γ or the trivial partition. Since $Q(4) = 1/4$ lies in the null range $[1/10, 2/5]$, neither is informative in this model. Retaining only a common summary cannot recover the desired growth. The model matters: with the same null segment and alternative $Q' = \frac{1}{10}(1,1,1,7)$, the coarse scores $e(\{1,2,3\}) = 1/2$ and $e(\{4\}) = 7/4$ are valid and have positive growth. Lifting them to both settings works because every null top-bin probability is at most $2/5$, and $(1/2)^3(7/4)^7 > 1$.

Passing the setting to the accumulator permits separate score tables. Retaining history permits the pooled rule in Section 7, whose successive ratios need not be calibrated one-step factors. It remains valid against each fixed null law; Corollary 19 gives its failure under joint changes of null law and protocol.

The example also separates the present problem from robust probability updating (van Ommen et al., 2016). Both problems ask for a rule that tolerates uncertainty about reporting. In robust updating, the outcome marginal is fixed, the reporter can choose a truth-preserving message after seeing the outcome, and the recipient predicts a distribution over outcomes. The graph and matroid classifications of van Ommen et al. (2016) concern reporting strategies that remain optimal across specified symmetric loss functions. Here the partition is chosen before the outcome, the scientific null contains several distributions, and the required output is a scalar factor with a null expectation bound and positive alternative growth. The obstruction has a graph message structure because every message contains at most two outcomes. Its failure of common growth therefore concerns a different requirement from loss-invariant probability updating.

4.2 A sharp boundary for randomized truthful reports

The deterministic restriction matters for randomized reports. A finite *coarsening-at-random* (CAR) channel has nonnegative weights c_A, indexed by nonempty subsets of Ω, with

$$\sum_{A \ni i} c_A = 1 \quad (i \in \Omega), \qquad K(i,A) = c_A \mathbf{1}_{\{i \in A\}}.$$

It reports the subset label A with probability $K(i,A)$ when the outcome is i (Gill and Grünwald, 2008). The label is truthful: it contains the outcome. The actual report event is $\{\text{output} = A\}$ on the outcome space enlarged by the reporting coin. Its containment in $\{X \in A\}$ can be strict. This extension requires equal scores for equal subset labels across channels.

The report law is $T_K P(A) = c_A P(A)$. A channel is informative when $T_K Q \notin T_K \mathcal{C}$. On the union $Y_{\mathcal{K}}$ of the supported labels of a family $\mathcal{K}$, common validity and growth mean

$$\sum_{A \in Y_{\mathcal{K}}} c_A P(A) F(A) \le 1 \quad (P \in \mathcal{C},\ K \in \mathcal{K}), \qquad G_K(F) = \sum_{A : c_A > 0} c_A Q(A) \log F(A) > 0$$

for each informative K requiring growth. For a finite family, the same finite-dimensional separation proof as Proposition 3 applies to these linear report maps. Choosing a fixed partition before the outcome gives a CAR channel. The channels below also use the known triangle mechanism that reports each of $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ with CAR weight $1/2$ (Gill and Grünwald, 2008). The mechanism and CAR geometry are established; the result concerns their common-score constraint.

Theorem 5 (The CAR outcome threshold)**.** *On at most two outcomes, one positive subset-label rule is valid under every CAR channel and has positive log growth under each informative one. On*

three outcomes, two informative CAR channels and a positive null segment exist such that every common valid rule differing from one on the observed label alphabet has negative log growth under at least one channel. Three is therefore the smallest outcome count for this failure among CAR channels.

Proof. For $\Omega = \{1, 2\}$, the CAR equations force

$$c_{\{1\}} = c_{\{2\}} = t, \qquad c_\Omega = 1 - t, \qquad 0 \le t \le 1.$$

If $Q \notin \mathcal{C}$, Lemma 1 on the full partition supplies a positive valid f with $\mathbb{E}_Q \log f > 0$. Set $F(\{i\}) = f(i)$ and $F(\Omega) = 1$. Under every channel t, its null expectation is $t\mathbb{E}_P f + 1 - t \le 1$ and its log growth is $t\mathbb{E}_Q \log f$. A channel is informative exactly when $t > 0$ and $Q \notin \mathcal{C}$. If $Q \in \mathcal{C}$, no channel is informative and $F = 1$ suffices. One outcome is immediate. Positive growth is channel-specific; it need not have a positive lower bound over a family with $t \downarrow 0$.

For three outcomes take

$$Q = \tfrac{1}{3}(1, 1, 1), \quad P_0 = (1/12, 1/12, 5/6), \quad P_1 = (5/12, 1/2, 1/12), \quad \mathcal{C} = \operatorname{conv}\{P_0, P_1\}.$$

In label order $Y_\mathcal{K} = (\{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\})$, use

$$c^{(1)} = (1/2, 0, 1/4, 1/4, 3/4), \qquad c^{(2)} = (0, 1/2, 1/4, 3/4, 1/4).$$

Each vector satisfies the CAR equations. Each channel is a half mixture of the triangle mechanism and a binary partition: $\{\{1\}, \{2, 3\}\}$ for the first and $\{\{2\}, \{1, 3\}\}$ for the second. Both report maps are injective, since their three positive pair-label coordinates recover all three master probabilities. Also $Q \notin \mathcal{C}$: equality of the first coordinate in $(1 - \theta)P_0 + \theta P_1$ forces $\theta = 3/4$, where the second coordinate is $19/48 \neq 1/3$. Both channels are informative.

Write $P^{[j]} = T_{K_j} P$ and $Q^{[j]} = T_{K_j} Q$. Direct calculation gives

$$\begin{aligned} r &= \tfrac{95}{216} Q^{[1]} + \tfrac{121}{216} Q^{[2]} \\ &= \tfrac{5}{54} P_0^{[1]} + \tfrac{1}{3} P_1^{[1]} + \tfrac{13}{54} P_0^{[2]} + \tfrac{1}{3} P_1^{[2]} \\ &= (95/1296, 121/1296, 1/6, 229/648, 203/648). \end{aligned} \tag{10}$$

The null weights and the alternative weights each sum to one, and every coordinate of r is positive. Validity gives $\mathbb{E}_r F \le 1$. For positive F, Jensen's inequality gives

$$\tfrac{95}{216} G_{K_1}(F) + \tfrac{121}{216} G_{K_2}(F) = \mathbb{E}_r \log F \le \log \mathbb{E}_r F \le 0.$$

Equality requires a constant rule on $Y_\mathcal{K}$ and $\mathbb{E}_r F = 1$, so the constant is one. A zero score gives growth $-\infty$ under a channel assigning that label positive alternative probability. Every common valid rule other than one therefore has negative growth under at least one channel. □

5 Classification of protocol pairs

The counterexample shows that some reporting designs fail for some models. A designer may instead want a structural guarantee that holds whatever the positive model turns out to be: a list of the pairs that admit a common growing table whenever both protocols are informative. Such a guarantee removes the need to check joint feasibility anew to establish existence, although the score itself still depends on the model. The next theorem gives this list and constructs an obstructing model for every pair outside it.

Write $\Pi \preceq \Sigma$ when every cell of Π lies within a cell of Σ; Π is the finer partition.

Definition 6 (Universal compatibility of a pair)**.** A pair is universally compatible if every positive Q and compact convex positive null class $\mathcal{C}$ making both protocols informative admits a positive common rule that is valid and grows under both. The rule may depend on the model.

Theorem 7 (Pair classification)**.** *Two deterministic partitions are universally compatible if and only if*

$$\Pi \preceq \Sigma \quad or \quad \Sigma \preceq \Pi \quad or \quad \Pi \cap \Sigma = \varnothing. \tag{11}$$

The last condition means that the partitions have no identical cells. For every incomparable pair sharing a cell, a positive null segment and positive alternative make both protocols informative while preventing simultaneous positive growth.

The converse must realize the conflicting report laws within one model on Ω. The construction uses shared-cell mass to force any matching null to the segment's midpoint. Two further perturbations each preserve one protocol's probabilities and alter the other's, keeping both protocols informative there. Choosing a different endpoint for each protocol then gives the obstructing report-law mixture.

Proof. If the two report alphabets are disjoint, their separate optima define a common rule on the union. If $\Pi \preceq \Sigma$, take a positive-growth valid score on the informative coarser protocol Σ. Give each Π-cell the score of its containing Σ-cell. This preserves every null expectation and the alternative growth, and it agrees on shared cells. The other refinement direction is symmetric.

For the converse, suppose the pair is incomparable and shares at least one cell. Let S be the union of all shared cells, choose any positive Q, and set $s = Q(S)$. The shared cell and positivity of Q give $s > 0$; incomparability gives $s < 1$, since $S = \Omega$ would force $\Pi = \Sigma$. There are zero-sum vectors h, k, supported outside S, satisfying

$$T_\Pi h = 0, \quad T_\Sigma h \neq 0, \qquad T_\Sigma k = 0, \quad T_\Pi k \neq 0. \tag{12}$$

To obtain h, choose two atoms in one Π-cell and different Σ-cells and subtract their coordinate unit vectors. Such atoms exist because Π does not refine Σ. Their Π-cell cannot be a shared cell, so they lie outside S. Reverse the protocols to obtain k.

Define the zero-sum vector

$$v_i = \begin{cases} Q(i)/s, & i \in S, \\ -Q(i)/(1-s), & i \notin S. \end{cases}$$

Choose $0 < \delta < \min(s, 1-s)$ and a sufficiently small $\varepsilon > 0$. Set

$$P_0 = Q + \delta v + \varepsilon h, \qquad P_1 = Q - \delta v + \varepsilon k, \qquad \mathcal{C} = \operatorname{conv}\{P_0, P_1\}. \tag{13}$$

The distributions remain positive. If the projection of $(1-\theta)P_0 + \theta P_1$ equalled the projected alternative under either protocol, its mass on S would force $\theta = 1/2$. At that point the Π-projection differs by $(\varepsilon/2)T_\Pi k$ and the Σ-projection differs by $(\varepsilon/2)T_\Sigma h$. Both differences are nonzero. Both protocols are informative.

Put $\lambda = \frac{1}{2}[1 - \delta/(1-s)] \in (0,1)$. On the common report alphabet,

$$\lambda Q^{[\Pi]} + (1-\lambda)Q^{[\Sigma]} = \tfrac{1}{2}P_0^{[\Pi]} + \tfrac{1}{2}P_1^{[\Sigma]}. \tag{14}$$

On a private Π-cell, $P_0(A) = [1 - \delta/(1-s)]Q(A)$ by (12). On a private Σ-cell, $P_1(A) = [1 + \delta/(1-s)]Q(A)$. On a shared cell, the two changes in mass cancel in the average. These three cases prove the identity. Its right side gives a calibration bound and its left side is an alternative mixture with two positive weights. Jensen's inequality excludes positive growth under both protocols. □

The converse constructs the null class on the original outcome space. The report-space separation criterion alone does not provide that realization. An incomparable pair with a shared report can still admit a common positive-growth rule for a particular model. For example, with a simple null P, the event rule $Q(A)/P(A)$ is valid under every partition and grows under each informative one. The theorem quantifies over models and classifies pairs. Section 5.1 shows that the criterion applied pairwise does not extend to larger families.

5.1 Families of protocols

Call a family of protocols universally compatible if every positive Q and compact convex positive null class $\mathcal{C}$ making all its members informative admits a positive common rule that is valid and grows under every member. For a pair this is the definition above. Condition (11), imposed on every pair of a family, does not give universal compatibility of the family.

Proposition 8 (Pairwise compatibility does not extend to families)**.** *Let* $Q = \frac{1}{4}(1,1,1,1)$ *and* $\mathcal{C} = \mathrm{conv}\{R_0, R_1\}$*, where*

$$R_0 = (1/5, 2/5, 3/10, 1/10), \qquad R_1 = (1/5, 1/10, 1/10, 3/5),$$

and let $\mathcal{V} = \mathcal{H} = \{\Pi_1, \Pi_2, \Pi_3\}$*, where*

$$\Pi_1 = \{\{1\},\{2\},\{3\},\{4\}\}, \qquad \Pi_2 = \{\{1\},\{2,3,4\}\}, \qquad \Pi_3 = \{\{1,2\},\{3\},\{4\}\}.$$

Every pair among Π_1, Π_2, Π_3 *satisfies* (11)*, and all three protocols are informative. Every common valid rule* F *satisfies*

$$\tfrac{2}{25} G_{\Pi_1}(F) + \tfrac{8}{25} G_{\Pi_2}(F) + \tfrac{3}{5} G_{\Pi_3}(F) \le 0. \tag{15}$$

No common valid rule grows under all three protocols, so the family is not universally compatible.

Proof. The protocol Π_1 refines Π_2 and Π_3, and Π_2 and Π_3 share no cell, so every pair satisfies (11). Every $P \in \mathcal{C}$ has $P(1) = 1/5$, while $Q(1) = 1/4$; this makes Π_1 and Π_2 informative. Under Π_3, the segment $(1-\theta)R_0 + \theta R_1$ meets $P(\{1,2\}) = 1/2$ only at $\theta = 1/3$, where $P(\{3\}) = 7/30$ differs from $Q(\{3\}) = 1/4$; this makes Π_3 informative.

Use the alphabet $Y = (\{1\},\{2\},\{3\},\{4\},\{2,3,4\},\{1,2\})$. Direct calculation gives

$$\begin{aligned} r &= \tfrac{2}{25} Q^{[\Pi_1]} + \tfrac{8}{25} Q^{[\Pi_2]} + \tfrac{3}{5} Q^{[\Pi_3]} \\ &= \tfrac{1}{5} R_1^{[\Pi_1]} + \tfrac{3}{10} R_0^{[\Pi_2]} + \tfrac{1}{2} R_0^{[\Pi_3]} \\ &= (1/10, 1/50, 17/100, 17/100, 6/25, 3/10). \end{aligned} \tag{16}$$

The vector r lies in $N_{\mathcal{V}} \cap A_{\mathcal{H}}$. Proposition 3 and Jensen's inequality give (15) for every common valid rule: validity gives $\mathbb{E}_r F \le 1$, and the weighted growth is at most $\log \mathbb{E}_r F$. A zero score gives $-\infty$ under at least one protocol. □

The positive lifting argument also gives a sufficient condition for larger families. Define the *sharing graph* of a finite family $\mathcal{F}$ by joining two partitions when they have an identical cell. The components separate the score coordinates that can be coupled.

Proposition 9 (A common coarsening in each component)**.** *Suppose each component* $\mathcal{F}_c$ *of the sharing graph contains a member* Γ_c *such that* $\Pi \preceq \Gamma_c$ *for every* $\Pi \in \mathcal{F}_c$*. Then* $\mathcal{F}$ *is universally compatible.*

Proof. Fix a model making every member informative. Lemma 1 supplies a positive valid score e_c with positive growth on Γ_c. For a cell A of any $\Pi \in \mathcal{F}_c$, let $B_c(A)$ be the unique Γ_c-cell containing A, and set $F(A) = e_c(B_c(A))$. A shared cell has the same containing cell regardless of which partition reports it. Distinct components share no score coordinate, so the component rules define one function. For every $P \in \mathcal{C}$ and $\Pi \in \mathcal{F}_c$,

$$\sum_{A \in \Pi} P(A)F(A) = \sum_{B \in \Gamma_c} P(B)e_c(B) \leq 1, \qquad G_\Pi(F) = \sum_{B \in \Gamma_c} Q(B) \log e_c(B) > 0.$$

These are the required validity and growth inequalities. □

The condition is not necessary. On $\Omega = \{1, 2, 3\}$, consider

$$\Pi_0 = \{\{1\}, \{2\}, \{3\}\}, \quad \Pi_1 = \{\{1\}, \{2, 3\}\}, \quad \Pi_2 = \{\{2\}, \{1, 3\}\}.$$

The sharing graph is connected through Π_0, and no member is coarser than both binary partitions. Yet Theorem 13 gives universal compatibility, and Corollary 14 extends this family to every outcome count. Proposition 8 gives a connected four-outcome family without such a member that is obstructed.

Adding an informative common coarsening can restore universal compatibility. For an incomparable pair Π, Σ sharing a cell, Theorem 7 supplies an obstructing model. Every common coarsening is uninformative in that model: its positive-growth score would otherwise lift to the pair. If a common coarsening Γ is added to the family and required to be informative, its score lifts to all three members by the same lifting calculation. A shared cell A always supplies a nontrivial choice $\Gamma = \{A, \Omega \setminus A\}$. Universal compatibility can therefore change when a new informativeness requirement removes obstructing models.

Universal compatibility is not monotone in the family. The family $\{\Pi, \Sigma, \Gamma\}$ with $\Gamma = \{A, \Omega \setminus A\}$ is universally compatible by Proposition 9; removing Γ leaves the obstructed pair. Adding a member can also destroy universal compatibility: the partitions Π_2 and Π_3 of Proposition 8 share no cell and form a universally compatible pair by Theorem 7, while adding Π_1 obstructs the family.

A necessary condition comes from refinement. Informativeness passes from a partition to every partition that refines it, and a common rule for a family restricts to a common rule for any subfamily.

Lemma 10 (Obstructions pass to refining families)**.** *Let $\mathcal{F}' \subseteq \mathcal{F}$ be finite families of partitions such that every member of $\mathcal{F}$ refines some member of $\mathcal{F}'$. If $\mathcal{F}'$ is not universally compatible, then $\mathcal{F}$ is not universally compatible.*

Proof. Fix a positive model that makes every member of $\mathcal{F}'$ informative and in which no positive common rule is valid and grows under every member of $\mathcal{F}'$. Let $\Gamma \preceq \Pi$ with Π informative. If some $P \in \mathcal{C}$ satisfied $T_\Gamma P = T_\Gamma Q$, then every Π-cell, being a union of Γ-cells, would satisfy $P(A) = Q(A)$, and Π would be uninformative. Every member of $\mathcal{F}$ refines an informative member of $\mathcal{F}'$ and is therefore informative in this model. A positive common rule that is valid and grows under every member of $\mathcal{F}$ would be valid and grow under every member of $\mathcal{F}'$. No such rule exists, so $\mathcal{F}$ is not universally compatible. □

Call a member of a finite family *maximal* if no other member is strictly coarser than it. Refinement is a partial order on the family, so every member refines a maximal member, and two distinct maximal members are incomparable.

Corollary 11 (Maximal members)**.** *If a finite family is universally compatible, then the set of its maximal members is universally compatible. In particular, a family with exactly two maximal members that share a cell is not universally compatible.*

Proof. Apply Lemma 10 with $\mathcal{F}'$ the set of maximal members. Two distinct maximal members are incomparable, so if they share a cell, Theorem 7 shows that the pair is not universally compatible. □

The necessary condition is not sufficient. The three-outcome family above and the family of Proposition 8 each have exactly two maximal members, and these share no cell. Theorem 7 makes each set of maximal members universally compatible, so both families meet the condition of Corollary 11; the first family is universally compatible and the second is not.

Two questions remain open. Call a family an *antichain* if none of its members refines another. The first question asks whether every antichain containing two members with a shared cell fails universal compatibility. Theorem 7 gives a positive answer for two members. The model of Theorem 4 gives a positive answer for the antichain $\{\Sigma_1, \Sigma_2, \{\{1,2\},\{3,4\}\}\}$. Its third member is incomparable with Σ_1 and Σ_2 and shares no cell with either. That member is informative in the model because $R_0(\{1,2\}) = 9/20$ and $R_1(\{1,2\}) = 19/60$ both lie below $Q(\{1,2\}) = 1/2$, and Theorem 4 excludes common growth under Σ_1 and Σ_2. The segment construction in the proof of Theorem 7 does not settle the general case, because it need not keep a third member informative. A positive answer would show, through Corollary 11, that every family whose maximal members include two with a shared cell fails universal compatibility.

The second question asks which families whose maximal members pairwise share no cell are universally compatible. The families covered by Proposition 9, whose maximal members are coarsest members of distinct sharing components, and those of Corollary 14 are universally compatible members of this class; the family of Proposition 8 is an obstructed member. A positive answer to the first question would reduce the classification of finite families to the second.

Remark (Refinement and comparison of experiments). Blackwell comparison orders information channels through performance in decision problems. A finite channel K simulates L by garbling when $L = KB$ for a row-stochastic matrix B (Blackwell, 1953; Bertschinger and Rauh, 2014). For the full-outcome deterministic channels $K_\Pi(i, A) = \mathbf{1}_{\{i \in A\}}$, $K_\Sigma = K_\Pi B$ holds exactly when $\Pi \preceq \Sigma$. Refinement gives the map sending each fine cell to its containing coarse cell. Conversely, a fixed Π-cell selects one row of B; that row must be the point mass at the Σ-cell containing each of its outcomes, so all those target cells coincide. This explains the lifting branch of Theorem 7. The disjoint-alphabet branch requires no information ordering because its score entries can be chosen separately.

6 The sharp bound for deterministic partitions

The pair classification guarantees a score for each eligible pair. A single score for an entire family must satisfy all its constraints together; separately chosen pairwise scores need not agree. I therefore need a simultaneous construction to establish the smallest possible outcome space for deterministic-partition failure. With three outcomes, every partition other than full reporting and the whole-space report consists of one singleton and one grouped pair. Once the singleton score is fixed, calibration bounds the score assigned to the pair. The following construction takes the smallest such bound over the null class.

Lemma 12 (A small-space extension). *Suppose $f(i) > 0$, $\mathbb{E}_P f \le 1$ for all $P \in \mathcal{C}$, and $\mathbb{E}_R f = 1$ for some $R \in \mathcal{C}$. For every nonempty $A \subseteq \Omega$, define*

$$U_f(A) = \min_{P \in \mathcal{C}} \frac{1 - \sum_{i \notin A} P(i) f(i)}{P(A)}. \tag{17}$$

This function is positive, agrees with f on singletons, and satisfies $U_f(\Omega) = 1$. For every m, it is valid under every partition consisting of one nonsingleton cell and the singletons outside that cell. If $m \leq 3$, every deterministic partition is full, trivial, or of that form, so U_f is valid under every deterministic partition.

Proof. Put $\Delta_P = 1 - \mathbb{E}_P f \geq 0$. The numerator equals $\sum_{i \in A} P(i) f(i) + \Delta_P > 0$. Compactness gives a positive attained minimum. For a singleton, every ratio is at least $f(i)$ and the active null R attains equality. The whole-space value is one. For a partition consisting of a nonsingleton A and its outside singletons, (17) applied at any $P \in \mathcal{C}$ gives

$$P(A) U_f(A) + \sum_{i \notin A} P(i) f(i) \leq 1.$$

This holds for every m. The full and trivial cases already satisfy validity, and on at most three outcomes every partition is full, trivial, or of the displayed form. □

Theorem 13 (Three outcomes permit a common rule)**.** *If $m \leq 3$, there is a positive event rule, defined on every nonempty event, that is valid under all deterministic partitions and grows under every informative one. Four outcomes are therefore necessary for a deterministic-partition obstruction with informative performance protocols. Theorem 4 attains this minimum with two protocols and one shared report; one protocol or no shared report cannot obstruct common growth.*

Proof. If $Q \in \mathcal{C}$, no partition is informative and $F = 1$ suffices. Otherwise set

$$l_i = \min_{P \in \mathcal{C}} P(i), \quad u_i = \max_{P \in \mathcal{C}} P(i), \quad I_+ = \{i : Q(i) > u_i\}, \quad I_- = \{i : Q(i) < l_i\}.$$

I construct h with $\sup_{P \in \mathcal{C}} \mathbb{E}_P h = 0$, $\mathbb{E}_Q h > 0$, positive coordinates on I_+, and negative coordinates on I_-. If $I_+ \neq \varnothing$, use $h = \mathbf{1}_{I_+} - u(I_+)$, where $u(B) = \max_{P \in \mathcal{C}} P(B)$. The inequality

$$Q(I_+) > \sum_{i \in I_+} u_i \geq u(I_+)$$

gives positive alternative expectation. The set I_+ is proper, so positivity of the nulls gives $0 < u(I_+) < 1$ and the required coordinate signs. If $I_+ = \varnothing$ and $I_- \neq \varnothing$, use $h = l(I_-) - \mathbf{1}_{I_-}$, where $l(B) = \min_{P \in \mathcal{C}} P(B)$ and

$$Q(I_-) < \sum_{i \in I_-} l_i \leq l(I_-).$$

If both sets are empty, strict separation of Q and $\mathcal{C}$, shifted to null supremum zero, gives h. Compactness supplies an active null R.

For small $\varepsilon > 0$, let $f_\varepsilon(i) = 1 + \varepsilon h_i$ and $F_\varepsilon = U_{f_\varepsilon}$. Lemma 12 gives positivity and validity under every partition. Full-report growth has derivative $\mathbb{E}_Q h > 0$ at zero.

For any m and $i \in \Omega$, the partition $\Lambda_i = \{\{i\}, \Omega \setminus \{i\}\}$ is informative exactly when $Q(i) > u_i$ or $Q(i) < l_i$, because the projection of the convex null onto that coordinate is the interval $[l_i, u_i]$. For $m = 3$, every informative partition other than the full one is some Λ_i. Substituting f_ε into the extension formula gives the complementary score as the minimum of $1 - \varepsilon p_i h_i / (1 - p_i)$ over $p_i \in [l_i, u_i]$. The ratio $p_i/(1 - p_i)$ increases with p_i, so the minimum uses u_i when $h_i > 0$ and l_i when $h_i < 0$. If $h_i = 0$, the complementary score equals one. For $h_i \neq 0$,

$$F_\varepsilon(\Omega \setminus \{i\}) = 1 - \varepsilon \frac{c_i h_i}{1 - c_i}, \qquad c_i = \begin{cases} u_i, & h_i > 0, \\ l_i, & h_i < 0. \end{cases}$$

Every informative isolated coordinate has $h_i \neq 0$, and its binary growth derivative is

$$\frac{h_i[Q(i) - c_i]}{1 - c_i} > 0.$$

The chosen signs ensure this inequality at every informative isolated coordinate. With two outcomes, the only nontrivial partition is full; with one outcome, none is informative. Since there are finitely many partitions, one sufficiently small positive ε gives positive growth under every informative partition.

For an explicit bound, write $F_\varepsilon(A) = 1 + \varepsilon d_A$. The coefficients d_A do not depend on positive ε. Let

$$B = \max_A |d_A|, \qquad a = \min_{\Pi \text{ informative}} \sum_{A \in \Pi} Q(A) d_A > 0.$$

When informative partitions exist, $B > 0$. Taking

$$0 < \varepsilon \leq \min\{(2B)^{-1}, a/(2B^2)\}$$

gives growth at least $\varepsilon a/2$ by $\log(1 + x) \geq x - x^2$ for $|x| \leq 1/2$.

Finally, one informative protocol has a positive separate optimum by Lemma 1. With no shared report, separate optima impose no equality constraints on one another. These observations prove the two remaining minimum counts. □

The lower bound depends on the model. It does not stay positive uniformly as alternatives approach the null. Uninformative partitions must remain valid, but the theorem imposes no nonnegative-growth requirement on them.

The same construction settles, on every outcome count, a family that Proposition 9 does not cover. Call Λ_i the *co-singleton* partition of i.

Corollary 14 (Co-singleton families on any outcome count)**.** *Let $m \geq 3$, let Π_{full} be the partition of Ω into singletons, and let $\Lambda_i = \{\{i\}, \Omega \setminus \{i\}\}$ for $i \in \Omega$. For every positive Q and compact convex positive $\mathcal{C}$ with $Q \notin \mathcal{C}$, some positive event rule is valid under Π_{full} and under every Λ_i, and it grows under each informative member of this collection. In particular, the family $\{\Pi_{\text{full}}\} \cup \{\Lambda_i : i \in J\}$ is universally compatible for every $J \subseteq \Omega$. For $|J| \geq 2$, its sharing graph is connected and no member is coarser than every other member, so Proposition 9 does not apply to it.*

Proof. Take h, f_ε, and $F_\varepsilon = U_{f_\varepsilon}$ from the proof of Theorem 13. The construction of h there uses only $Q \notin \mathcal{C}$, and it supplies an active null R with $\mathbb{E}_R f_\varepsilon = 1$. Validity under Π_{full} is $\mathbb{E}_P f_\varepsilon = 1 + \varepsilon \mathbb{E}_P h \leq 1$. Lemma 12 gives positivity and validity under every Λ_i, since Λ_i consists of the nonsingleton cell $\Omega \setminus \{i\}$ and the singleton outside it. Growth under Π_{full} has derivative $\mathbb{E}_Q h > 0$ at $\varepsilon = 0$. If Λ_i is informative, then $Q(i) \notin [l_i, u_i]$, so $i \in I_+ \cup I_-$ and h_i has the sign fixed in the proof of Theorem 13. The binary growth derivative $h_i[Q(i) - c_i]/(1 - c_i)$ computed there is then positive. The collection has $m + 1$ members, so one sufficiently small $\varepsilon > 0$ gives positive growth under every informative member; the explicit bound in the proof of Theorem 13 applies with B and a taken over this collection. A model that makes every member of $\{\Pi_{\text{full}}\} \cup \{\Lambda_i : i \in J\}$ informative has $Q \notin \mathcal{C}$, which proves universal compatibility.

Finally, Π_{full} shares $\{i\}$ with each Λ_i, so the sharing graph is connected. For $i \neq j$, the cell $\Omega \setminus \{j\}$ contains i and, since $m \geq 3$, another outcome, so it lies in no cell of Λ_i. The partitions Λ_i and Λ_j are therefore incomparable. They share no cell, because their singleton cells differ, their other cells differ, and each other cell contains at least two outcomes. The cell $\Omega \setminus \{i\}$ lies in no singleton, so Π_{full} is coarser than no Λ_i. □

Lemma 12 also gives validity under every partition with one nonsingleton cell and singletons elsewhere, but the corollary does not extend to that class. On four outcomes, Proposition 8 obstructs the full partition together with $\Lambda_1 = \{\{1\}, \{2,3,4\}\}$ and $\{\{1,2\}, \{3\}, \{4\}\}$; the obstruction concerns growth.

6.1 Finite reports on a countable outcome space

The finite mathematics depends on the distinctions the retained protocols can report. A countable master space creates no additional static difficulty when only finitely many report cells occur.

Proposition 15 (Exact reduction for finite reports)**.** *Let Ω be a nonempty countable set, let $\mathcal{F}$ be a finite nonempty family of finite partitions, and let $\mathcal{B}$ be their common refinement, consisting of all nonempty intersections of one cell from each partition. Suppose Q and every $P \in \mathcal{C}$ have positive point probabilities, and $\mathcal{C}$ is nonempty, convex, and compact in total variation. Replacing each atom $B \in \mathcal{B}$ by one outcome preserves common-rule validity, every expected log score, and informativeness. The family is universally compatible on Ω exactly when its induced family on $\mathcal{B}$ is universally compatible. In particular, the pair classification and the component condition apply, and $|\mathcal{B}| \leq 3$ suffices for common growth across all informative members.*

Proof. The set $\mathcal{B}$ is a finite partition. Every report A is the union of its atoms, so the map

$$A \longmapsto \overline{A} = \{B \in \mathcal{B} : B \subseteq A\}$$

is injective and preserves identical reports and refinement. The linear map $L(P) = (P(B))_{B \in \mathcal{B}}$ is continuous in total variation. Its image $\overline{\mathcal{C}} = L(\mathcal{C})$ is compact and convex. Every atom is nonempty and has positive probability under every master law. Compactness supplies a positive lower bound on the finitely many atom coordinates over $\overline{\mathcal{C}}$. The alternative $\overline{Q} = L(Q)$ is also positive.

Set $\overline{F}(\overline{A}) = F(A)$. For every retained partition, its report probabilities are unchanged, so

$$\sum_{A \in \Pi} P(A)F(A) = \sum_{\overline{A} \in \overline{\Pi}} \overline{P}(\overline{A})\overline{F}(\overline{A}), \qquad G_{\Pi}(F) = G_{\overline{\Pi}}(\overline{F}).$$

The projected null class and alternative report law are likewise unchanged. This proves the model-specific reduction.

For the universal assertion, every positive finite model on $\mathcal{B}$ also lifts to Ω. Fix positive conditional weights $w_B(i)$ summing to one within each atom B, and define $U(p)(i) = p(B)w_B(i)$ for $i \in B$. Then $L \circ U$ is the identity and $\|U(p) - U(q)\|_1 = \|p - q\|_1$. Positive alternatives and compact convex positive null classes lift with exactly the same report laws. Universal compatibility therefore holds in either setting exactly when it holds in the other. Apply the finite results on $\mathcal{B}$. □

This reduction covers only the retained finite reports and partitions formed from their atoms. It does not cover a protocol that splits an atom or has infinitely many cells. No probability law on a countably infinite set has a common positive lower bound on all point probabilities: such a bound c would imply $1 \geq nc$ for every n. Infinite report alphabets require separate control of compactness, separation, and log-score integrability.

7 Sequential use and the boundary of the conclusion

Repeated use requires a distinction between multiplying common calibrated factors and assigning one score to a whole report history.

7.1 Multiplying common factors

Conditional calibration, rather than separate marginal expectation bounds, justifies multiplication. Let $\mathcal{G}_t$ contain all information available before X_t, including the current rule and reporting protocol, and let $\mathcal{F}_{t-1} \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t$. If the conditional master law given $\mathcal{G}_t$ is some $P_t \in \mathcal{C}$ and F_t is a common valid rule, then

$$W_n = \prod_{t=1}^{n} F_t(A_t), \qquad \mathbb{E}[W_t \mid \mathcal{G}_t] \le W_{t-1}.$$

The tower property gives the supermartingale inequality relative to $\mathcal{F}_{t-1}$. This standard conditional argument permits predictable changes in null distributions and protocols (Grünwald et al., 2024).

An e-process is a nonnegative adapted process with null expectation at most one at every almost surely finite stopping time in its specified filtration. An *adjuster* is a nondecreasing function $A : [1, \infty] \to [0, \infty]$ satisfying $\int_1^\infty A(x)x^{-2}\,dx \le 1$. When both initial sigma-fields are trivial, filtration lifting converts an e-process starting at one into an e-process for a larger filtration by applying an adjuster to its running maximum and setting the lifted process to one at time zero (Choe and Ramdas, 2026). The present question adds equality of factors on shared reports and asks for positive growth under each retained protocol.

In the laboratory example, the analyst announces a common valid table from past information. The technician observes it, selects the reporting setting, and then takes the measurement.

Corollary 16 (A responsive reporting policy)**.** *In the two-protocol example, suppose the scorer chooses a common valid rule from past information, and the reporter observes the realized rule before choosing the current protocol. For every such scoring policy, a predictable reporting policy makes W a nonnegative supermartingale under conditional master law Q. The product converges almost surely to a finite limit.*

Proof. Identity (9) implies

$$\tfrac{9}{20}\mathbb{E}_{Q^{[1]}} F_t + \tfrac{11}{20}\mathbb{E}_{Q^{[2]}} F_t \le 1.$$

Choose a protocol whose alternative expectation is at most one, with deterministic tie-breaking. The reporter chooses before the current outcome. The conditional product inequality follows, and the nonnegative supermartingale convergence theorem gives a finite limit. □

The reporter must observe the realized rule, including the result of any private randomization, and respond before the outcome. The corollary does not cover a scorer who changes the rule after learning the current protocol. It also does not say that every fixed protocol defeats every adaptive scorer.

7.2 Scoring joint histories

The responsive-policy result requires each update to use a common conditionally valid factor. A history score can preserve equality for identical reported histories without imposing that requirement on every successive ratio. Use the pre-outcome information $\mathcal{G}_t$ and post-report filtration $\mathcal{F}_t$ above, with the chosen partition Π_t measurable in $\mathcal{G}_t$ and its report A_t measurable in $\mathcal{F}_t$. Define

$$U_n = \frac{\prod_{t=1}^{n} Q(A_t)}{\sup_{P \in \mathcal{C}} \prod_{t=1}^{n} P(A_t)}, \qquad U_0 = 1. \tag{18}$$

This likelihood-envelope construction is established; its relation to e-processes is discussed by Chugg et al. (2026) and Koolen and Grünwald (2022). The next proposition states the guarantee under the alternative and the guarantee under each null separately, because the two use different conditional laws.

Proposition 17 (Pooled growth under predictable reporting)**.** *Let $\varnothing \neq \mathcal{H}$ be a finite family of informative protocols, and let each $\Pi_t \in \mathcal{H}$ be chosen with the measurability just specified.*

(a) Growth under the alternative. *If the conditional master law given $\mathcal{G}_t$ is Q at every time t, then, almost surely under the law that this reporting policy induces,*

$$\liminf_{n\to\infty} \frac{\log U_n}{n} \geq d > 0, \qquad d = \min_{\Pi\in\mathcal{H}} \min_{P\in\mathcal{C}} D(Q_\Pi \| P_\Pi).$$

(b) Validity under each fixed null. *If the conditional master law given $\mathcal{G}_t$ is one fixed $P \in \mathcal{C}$ at every time t, then $\mathbb{E}_P U_\tau \leq 1$ for every almost surely finite stopping time τ of $(\mathcal{F}_t)$.*

Part (a) holds for each reporting policy under its own induced law; it does not assert one probability-one event for all policies. Part (b) fixes the null law. Theorem 18 and Corollary 19 address the larger class of conditional null changes.

Proof. For (b), Π_t is $\mathcal{G}_t$-measurable and its cells partition Ω, so

$$\mathbb{E}_P\Big[\frac{Q(A_t)}{P(A_t)} \,\Big|\, \mathcal{G}_t\Big] = \sum_{A\in\Pi_t} P(A)\frac{Q(A)}{P(A)} = 1.$$

Since $\mathcal{F}_{t-1} \subseteq \mathcal{G}_t$, the tower property makes $M_n^P = \prod_{t\leq n} Q(A_t)/P(A_t)$ a nonnegative $\mathcal{F}_n$-martingale with $M_0^P = 1$. The pooled score satisfies $U_n \leq M_n^P$ at every time. Optional stopping at the bounded time $\tau \wedge n$ gives $\mathbb{E}_P U_{\tau\wedge n} \leq \mathbb{E}_P M_{\tau\wedge n}^P = 1$, and Fatou's lemma gives $\mathbb{E}_P U_\tau \leq 1$.

For (a), let $Y = \bigcup_{\Pi\in\mathcal{H}} \Pi$. For $A \in Y$, write

$$C_n(A) = \sum_{t=1}^n \mathbf{1}_{\{A_t=A\}}, \qquad N_n(A) = \sum_{t=1}^n \mathbf{1}_{\{A\in\Pi_t\}}.$$

The protocol is known before the current outcome, so

$$\mathbb{E}_Q[\mathbf{1}_{\{A_t=A\}} \mid \mathcal{G}_t] = Q(A)\mathbf{1}_{\{A\in\Pi_t\}}.$$

The tower property makes $Z_n(A) = C_n(A) - Q(A)N_n(A)$ an $\mathcal{F}_n$-martingale with increments bounded by one. The bounded martingale strong law gives $Z_n(A)/n \to 0$ almost surely, simultaneously for the finite alphabet Y. Compact positivity gives

$$B = \max_{P\in\mathcal{C}, A\in Y} \left|\log \frac{Q(A)}{P(A)}\right| < \infty.$$

For every $P \in \mathcal{C}$,

$$\begin{aligned}\sum_{t=1}^n \log \frac{Q(A_t)}{P(A_t)} &= \sum_{A\in Y} C_n(A) \log \frac{Q(A)}{P(A)} \\ &= \sum_{t=1}^n D(Q_{\Pi_t} \| P_{\Pi_t}) + \sum_{A\in Y} Z_n(A) \log \frac{Q(A)}{P(A)}.\end{aligned}$$

The last sum has absolute value at most $B\sum_{A\in Y} |Z_n(A)| = o(n)$, uniformly in P. Each protocol is informative, and divergence is continuous on its compact positive projected null class, so its minimum is positive. Finiteness of $\mathcal{H}$ gives $d > 0$. Taking the infimum over P in the last display yields $\log U_n \geq nd - o(n)$. □

If reporting choices are deterministic functions of previously reported history, a realized history identifies the rectangle $D = A_1 \times \cdots \times A_n$. Formula (18) equals the event score

$$Q^{\otimes n}(D) / \sup_{P \in \mathcal{C}} P^{\otimes n}(D).$$

The same formula is valid for any fixed partition of Ω^n by termwise comparison to its probability numerator. It preserves event identity at the history level.

For a fixed repeated protocol, choose a master $P^* \in \mathcal{C}$ with the projection from Lemma 1. Then

$$U_n \le \prod_{t=1}^{n} \frac{Q(A_t)}{P^*(A_t)},$$

and every factor on the right is valid under all $P_t \in \mathcal{C}$. The right side is a test supermartingale under conditional null changes. Joint changes of null law and protocol need a separate analysis.

7.3 Event histories under conditional null changes

An event-history score is one fixed sequence of maps $w_n : Y^n \to [0, \infty)$, with $W_n = w_n(A_1, \ldots, A_n)$ and $W_0 = 1$. The score observes neither protocol labels nor the reporter's auxiliary coins. Its report filtration is $\mathcal{R}_n = \sigma(A_1, \ldots, A_n)$.

A *drifting-null policy* chooses the pair $(P_t, \Pi_t) \in \mathcal{C} \times \mathcal{V}$ from past information and fresh randomization before drawing the current outcome. The pair and the randomization are measurable in the full pre-outcome information $\mathcal{G}_t$, where $\mathcal{F}_{t-1} \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t$, and the conditional master law is $\mathcal{L}(X_t \mid \mathcal{G}_t) = P_t$. An *alternative policy* chooses $\Pi_t \in \mathcal{H}$ in the same order and satisfies $\mathcal{L}(X_t \mid \mathcal{G}_t) = Q$. The report is the cell of Π_t containing X_t. Joint selection permits dependence between the null law and the protocol. The score maps remain the same under every policy.

Theorem 18 (History scores and conditional null changes)**.** *In the finite positive model, suppose every member of $\mathcal{H}$ is informative. The following are equivalent.*

(i) $N_{\mathcal{V}} \cap A_{\mathcal{H}} = \varnothing$.

(ii) There are a positive common rule F and a constant $\gamma > 0$ such that $W_n = \prod_{t=1}^{n} F(A_t)$ is an e-process in $(\mathcal{F}_t)$ under every drifting-null policy and

$$\liminf_{n \to \infty} n^{-1} \log W_n \ge \gamma$$

almost surely under every alternative policy.

(iii) There is a nonnegative event-history score with $\mathbb{E}W_n \le 1$ at every deterministic n under every drifting-null policy and $W_n \to \infty$ almost surely under every alternative policy.

Each almost-sure assertion refers to the law induced by the particular policy. The constant in (ii) is common to all alternative policies.

Proof. For (i) implies (ii), Proposition 3 supplies a positive common valid F with $\gamma = \min_{\Pi \in \mathcal{H}} G_\Pi(F) > 0$. Conditional calibration given $\mathcal{G}_t$ and the tower property make its product a nonnegative $(\mathcal{F}_t)$-supermartingale under every drifting-null policy. Optional stopping at bounded times and Fatou's lemma give the e-process bound. Under every alternative policy, the conditional expected log increment given $\mathcal{G}_t$ is $G_{\Pi_t}(F) \ge \gamma$. The centered log increments are bounded martingale

differences because Y is finite and F is positive. Their average converges to zero, which proves the stated growth bound. Statement (ii) implies (iii).

For the converse, suppose $r \in N_{\mathcal{V}} \cap A_{\mathcal{H}}$. Its convex-hull representations are finite:

$$r = \sum_{j=1}^{k} w_j P_j^{[\Pi_j]} = \sum_{\Pi \in \mathcal{H}} \lambda_\Pi Q^{[\Pi]}, \qquad \sum_j w_j = \sum_{\Pi \in \mathcal{H}} \lambda_\Pi = 1,$$

where the weights are nonnegative, $P_j \in \mathcal{C}$, and $\Pi_j \in \mathcal{V}$. No polytope assumption on $\mathcal{C}$ is needed. Construct a null policy by drawing independent pairs $(P_t, \Pi_t) = (P_j, \Pi_j)$ with probabilities w_j at each time, then drawing the outcomes independently conditional on these pairs, each from its selected P_j. This satisfies the conditional null law given $\mathcal{G}_t$, which includes the selected pair, and its reports are independent with law r. Under an alternative policy, draw independent protocols with probabilities λ_Π and independently draw every outcome from Q. Its reports are also independent with law r.

Both policies therefore induce $r^{\otimes \mathbb{N}}$ on the score's inputs. If (iii) held, $W_n \to \infty$ would hold under this null policy as well. Fatou's lemma would give

$$\infty = \mathbb{E} \liminf_n W_n \le \liminf_n \mathbb{E} W_n \le 1,$$

a contradiction. This proves (iii) implies (i). □

The necessity argument uses only deterministic-time validity. It requires growth under every alternative policy, including randomized protocol mixtures. Agreement of the two experiments on reported histories is sufficient; their protocol labels and auxiliary coins need not have the same joint law. For $\mathcal{V} = \mathcal{H}$ consisting of two protocols, Theorem 7 also classifies universal existence of these history guarantees. The theorem makes no impossibility claim under a fixed null law, where Proposition 17 still applies. Nor does it assert drift validity of the particular pooled score whenever the hulls are disjoint.

Corollary 19 (The pooled score can fail under null drift)**.** *In* (6)–(7)*, independently at every time choose the null pair*

$$(R_0, \Sigma_1),\ (R_1, \Sigma_1),\ (R_0, \Sigma_2),\ (R_1, \Sigma_2)$$

with probabilities $1/8, 3/8, 3/8, 1/8$*, respectively, before drawing the outcome from the selected endpoint. Under this drifting-null policy, the pooled score* (18) *satisfies*

$$n^{-1} \log U_n \longrightarrow d_r > 0 \quad \textit{almost surely},$$
$$d_r = \min_{P \in \mathcal{C}} \left\{ \tfrac{9}{20} D(Q_{\Sigma_1} \| P_{\Sigma_1}) + \tfrac{11}{20} D(Q_{\Sigma_2} \| P_{\Sigma_2}) \right\}.$$

For every $b > 1$*, the report-filtration stopping time* $\tau_b = \inf\{n : U_n \ge b\}$ *is almost surely finite and* $\mathbb{E} U_{\tau_b} \ge b > 1$*. Also* $\mathbb{E} U_n > 1$ *for some deterministic* n*.*

Proof. By (9), the reports are independent with law

$$r = (9/80, 9/40, 11/80, 11/40, 1/4) = \tfrac{9}{20} Q^{[1]} + \tfrac{11}{20} Q^{[2]}.$$

This is also the report law under Q with independent protocol probabilities $9/20, 11/20$. Let $\widehat{r}_n$ be the empirical report distribution. Compact positivity gives a finite bound B for all $|\log[Q(A)/P(A)]|$, with $P \in \mathcal{C}$ and $A \in Y$. The strong law and

$$\sup_{P \in \mathcal{C}} \left| \sum_{A \in Y} (\widehat{r}_n(A) - r(A)) \log \frac{Q(A)}{P(A)} \right| \le B \| \widehat{r}_n - r \|_1 \longrightarrow 0$$

give

$$n^{-1} \log U_n \longrightarrow \min_{P \in \mathcal{C}} \sum_{A \in Y} r(A) \log \frac{Q(A)}{P(A)} = d_r.$$

Both protocols are informative, so continuity on their compact positive projected null classes makes each minimum divergence strictly positive. Their positive weighted sum has a positive minimum. The growth limit makes every τ_b almost surely finite, and $U_{\tau_b} \geq b$ gives the stopped-expectation violation. If every deterministic-time expectation were at most one, Fatou's lemma and $U_n \to \infty$ would give a contradiction. □

The counterexample uses fixed pre-outcome selection probabilities and requires no response to the scorer. It preserves the fixed-null guarantee of Proposition 17 and the fixed-protocol drift bound above. Theorem 18 applies the established convexification and null-switching distinction (Zhang et al., 2024; Ramdas et al., 2022) to finite event histories; the certificate supplies its explicit master-space realization and the pooled-score failure.

8 What a common score commits its user to

Event agreement assigns one number to a shared report. Calibration bounds the expectation under every retained null experiment. Positive log growth makes a fixed rule accumulate under repeated draws from the stated alternative. The mathematics gives compatibility conditions for their conjunction. It does not decide whether an account of evidence should adopt the event-identity requirement; Sections 5 and 6 state which protocol pairs, and which of the studied families, can meet that requirement together with calibration and growth.

The score's specification must name the master model, the protocols requiring validity and growth, and the order in which the current rule and protocol are chosen. A finite polytope null permits the feasibility test (5). An obstruction identifies inconsistent scoring requirements; negative log growth supplies no posterior probability or empirical evidence against the alternative. A change in reporting that changes the master outcome law requires a different model, and outcome-dependent reporting requires a different calibration analysis.

Proposition 9 gives a direct sufficient design rule: each sharing component contains a retained member coarser than all its other members, and that member is required to be informative. Corollary 14 gives a second sufficient rule: the full-resolution protocol together with any set of co-singleton protocols, each of which reports only whether one specified outcome occurred. Corollary 11 gives a necessary rule: the maximal retained protocols must themselves form a universally compatible family, so a menu with exactly two maximal protocols that share a report is not universally compatible. Proposition 15 extends the static analysis to countable master spaces with finitely many retained reports. Theorem 5 shows that the minimum for failure falls from four outcomes for deterministic partitions to three for CAR channels, under equality of scores on truthful subset labels. The two deterministic-family questions, infinitely many report cells, and a structural classification of CAR families remain separate questions.

9 Conclusion

For two deterministic reporting partitions, universal compatibility of common calibrated positive-growth scores has an exact structural criterion: refinement comparability or absence of identical reports. Every remaining pair admits an informative positive model with no common valid rule growing under both. The explicit obstruction persists within a verified coordinate radius. For larger

families, pairwise compatibility is insufficient. A coarsest member in each sharing component is sufficient, as is full reporting with any set of co-singleton protocols. Compatibility of the maximal members is necessary but insufficient. The antichain question and the classification of families with pairwise unshared maximal members remain open. Four effective outcomes suffice and are necessary for deterministic-partition failure; at most three admit a simultaneous rule. CAR channels have a different sharp threshold: two outcomes always permit a common growing subset-label rule, while three can obstruct it. Finite retained reports give the static deterministic conclusions on a countable master space. Under conditional null changes, disjoint report hulls also characterize existence of event-history scores with validity and growth under every alternative protocol policy. A common product achieves a uniform positive exponential rate when they are disjoint. Pooled history scoring retains growth under alternative protocol changes and validity under each fixed null law, but the four-outcome certificate gives an allowed drifting-null policy under which it diverges and violates stopped and deterministic-time validity. These conclusions identify compatibility conditions for the stated scoring requirements; they do not establish practical demand for them.

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